Introduction to Mathematical Analysis I

Second Edition

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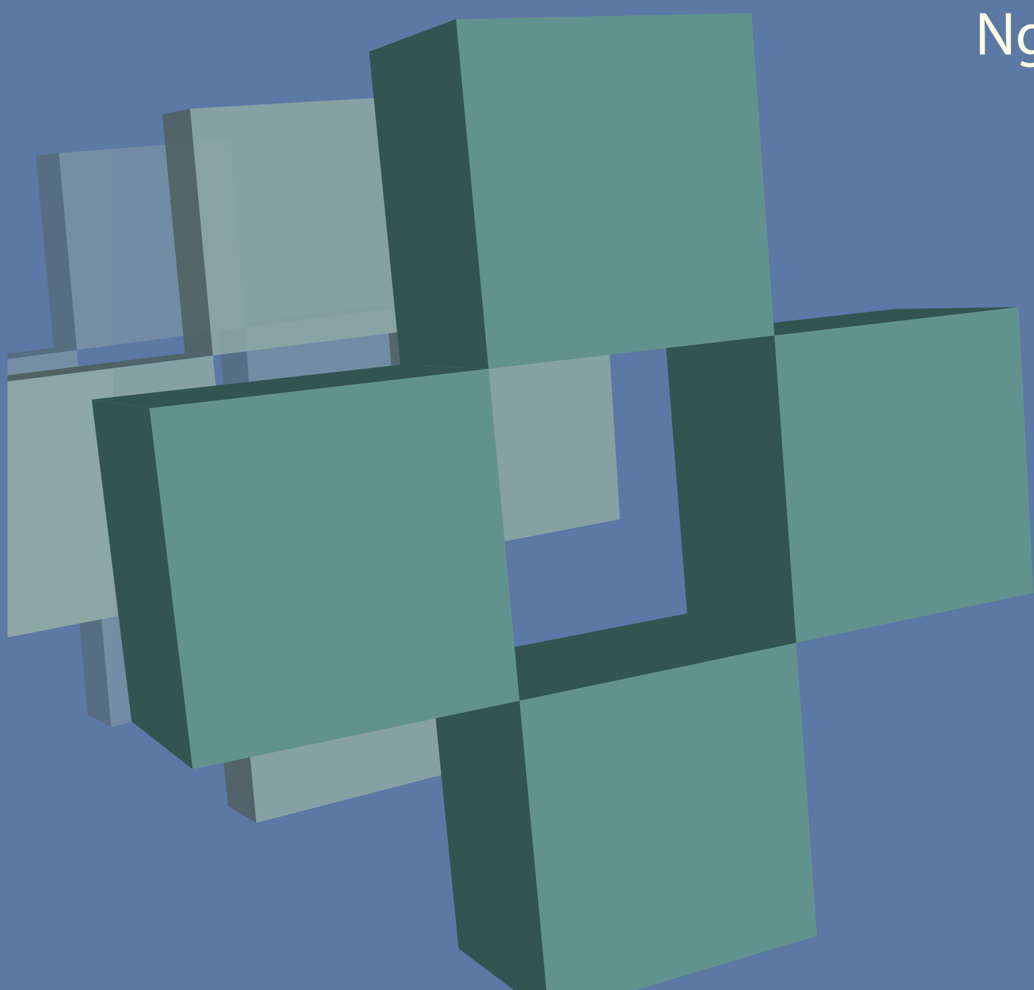

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Contents

Preface

Our goal in this set of lecture notes is to provide students with a strong foundation in mathematical analysis. Such a foundation is crucial for future study of deeper topics of analysis. Students should be familiar with most of the concepts presented here after completing the calculus sequence. However, these concepts will be reinforced through rigorous proofs.

The lecture notes contain topics of real analysis usually covered in a 10-week course: the completeness axiom, sequences and convergence, continuity, and differentiation. The lecture notes also contain many well-selected exercises of various levels. Although these topics are written in a more abstract way compared with those available in some textbooks, teachers can choose to simplify them depending on the background of the students. For instance, rather than introducing the topology of the real line to students, related topological concepts can be replaced by more familiar concepts such as open and closed intervals. Some other topics such as lower and upper semicontinuity, differentiation of convex functions, and generalized differentiation of non-differentiable convex functions can be used as optional mathematical projects. In this way, the lecture notes are suitable for teaching students of different backgrounds.

Hints and solutions to selected exercises are collected in Chapter 5. For each section, there is at least one exercise fully solved. For those exercises, in addition to the solutions, there are explanations about the process itself and examples of more general problems where the same technique may be used. Exercises with solutions are indicated by a ▶ and those with hints are indicated by a ▷.

Finally, to make it easier for students to navigate the text, the electronic version of these notes contains many *hyperlinks* that students can click on to go to a definition, theorem, example, or exercise at a different place in the notes. These hyperlinks can be easily recognized because the text or number is on a different color and the mouse pointer changes shape when going over them.

Changes in the Second Edition

The second edition includes a number of improvements based on recommendations from students and colleagues and on our own experience teaching the course over the last several years.

In this edition we streamlined the narrative in several sections, added more proofs, many examples worked out in detail, and numerous new exercises. In all we added over 50 examples in the main text and 100 exercises (counting parts).

We included more prominently the notion of compact set. We defined compactness as what is more commonly termed *sequential compactness*. Students find this definition easier to absorb than the general one in terms of open covers. Moreover, as the emphasis of the whole text is on sequences, this definition is easier to apply and reinforce.

The following are the more significant changes.

Chapter 1 Added the proofs of several properties of the real numbers as an ordered field.

Chapter 2 We added the proof that compactness is equivalent to closed and bounded in the main text.

Chapter 3 We added the theorem on extension of uniformly continuous functions and moved the discussion of Lipschiptz and Hölder continuous functions to the section on uniform continuity. We created a separate section for limit superior/inferior of functions.

Chapter 4 We clarified the statement and the proof of the second version of L'Hospital's rule.

We have used these notes several times to teach the one-quarter course Introduction to Mathematical Analysis I at Portland State University. As we are now preparing a companion text for the second term (Introduction to Mathematical Analysis II) we now added the roman numeral I to the title.

1. TOOLS FOR ANALYSIS

This chapter discusses various mathematical concepts and constructions which are central to the study of the many fundamental results in analysis. Generalities are kept to a minimum in order to move quickly to the heart of analysis: the structure of the real number system and the notion of limit. The reader should consult the bibliographical references for more details.

1.1 BASIC CONCEPTS OF SET THEORY

Intuitively, a set is a collection of objects with certain properties. The objects in a set are called the elements or members of the set. We usually use uppercase letters to denote sets and lowercase letters to denote elements of sets. If a is an element of a set A, we write $a \in A$. If a is not an element of a set A, we write $a \notin A$. To specify a set, we can list all of its elements, if possible, or we can use a defining rule. For instance, to specify the fact that a set A contains four elements a, b, c, d, we write

$$A = \{a, b, c, d\}.$$

To describe the set E containing all even integers, we write

$$E = \{x : x = 2k \text{ for some integer } k\}.$$

We say that a set A is a *subset* of a set B if every element of A is also an element of B, and write

$$A \subset B \text{ or } B \supset A.$$

Two sets are *equal* if they contain the same elements. If A and B are equal, we write $A = B$. The following result is straightforward and very convenient for proving equality between sets.

Theorem 1.1.1 Two sets A and B are equal if and only if $A \subset B$ and $B \subset A$.

If $A \subset B$ and A does not equal B, we say that A is a *proper* subset of B, and write

$$A \subsetneq B.$$

The set $\emptyset = \{x : x \neq x\}$ is called the *empty set*. This set clearly has no elements. Using Theorem 1.1.1, it is easy to show that all sets with no elements are equal. Thus, we refer to *the* empty set.

Throughout this book, we will discuss several sets of numbers which should be familiar to the reader:

- $\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of *natural numbers* or *positive integers*.
- $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$, the set of *integers* (that is, the natural numbers together with zero and the negative of each natural number).
- $\mathbb{Q} = \{m/n : m, n \in \mathbb{Z}, n \neq 0\}$, the set of *rational numbers*.
- $\mathbb{R}$, the set of *real numbers*.
- Intervals. For $a, b \in \mathbb{R}$, we have

$$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\},$$
$$(a,b] = \{x \in \mathbb{R} : a < x \leq b\},$$
$$[a,\infty) = \{x \in \mathbb{R} : a \leq x\},$$
$$(a,\infty) = \{x \in \mathbb{R} : a < x\},$$

  and similar definitions for (a,b), $[a,b)$, $(-\infty, b]$, and $(-\infty, b)$. We will say more about the symbols ∞ and $-\infty$ in Section 1.5.

Since the real numbers are central to the study of analysis, we will discuss them in great detail in Sections 1.4, 1.5, and 1.6.

For two sets A and B, the *union*, *intersection*, *difference*, and *symmetric difference* of A and B are given respectively by

$$A \cup B = \{x : x \in A \text{ or } x \in B\},$$
$$A \cap B = \{x : x \in A \text{ and } x \in B\},$$
$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}, \text{and}$$
$$A \,\Delta\, B = (A \setminus B) \cup (B \setminus A).$$

If $A \cap B = \emptyset$, we say that A and B are *disjoint*.

The difference of A and B is also called the *complement* of B in A. If X is a *universal set*, that is, a set containing all the objects under consideration, then the complement of A in X is denoted simply by A^c.

Theorem 1.1.2 Let A, B, and C be subsets of a universal set X. Then the following hold:

(a) $A \cup A^c = X$;

(b) $A \cap A^c = \emptyset$;

(c) $(A^c)^c = A$;

(d) *(Distributive law)* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;

(e) *(Distributive law)* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;

(f) *(DeMorgan's law)* $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$;

(g) *(DeMorgan's law)* $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$;

(h) $A \setminus B = A \cap B^c$.

Proof: We prove some of the results and leave the rest for the exercises.

(a) Clearly, $A \cup A^c \subset X$ since both A and A^c are subsets of X. Now let $x \in X$. Then either x is an element of A or it is not an element of A. In the first case, $x \in A$ and, so, $x \in A \cup A^c$. In the second case, $x \in A^c$ and, so, $x \in A \cup A^c$. Thus, $X \subset A \cup A^c$.

(b) No element of x can be simultaneously in A and not in A. Thus, $A \cap A^c = \emptyset$.

(d) Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Therefore, $x \in B$ or $x \in C$. In the first case, since x is also in A we get $x \in A \cap B$ and, hence, $x \in (A \cap B) \cup (A \cap C)$. In the second case, $x \in A \cap C$ and, hence, $x \in (A \cap B) \cup (A \cap C)$. Thus, in all cases, $x \in (A \cap B) \cup (A \cap C)$. This shows $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

Now we prove the other inclusion. Let $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$. In either case, $x \in A$. In the first case, $x \in B$ and, hence, $x \in B \cup C$. It follows in this case that $x \in A \cap (B \cup C)$. In the second case, $x \in C$ and, hence, $x \in B \cup C$. Again, we conclude $x \in A \cap (B \cup C)$. Therefore, $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$ as desired. □

A set whose elements are sets is often called a collection/family of sets and is often denoted by script letters such as $\mathscr{A}$ or $\mathscr{B}$.

Let I be a nonempty set such that to each $i \in I$ corresponds a set A_i. Then the family of all sets A_i as i ranges over I is denoted by

$$\{A_i : i \in I\}.$$

Such a family of sets is called an *indexed family* and the set I is called the *index set*. Consider the indexed family of sets $\{A_i : i \in I\}$. The union and intersection of this family as i ranges over I is defined respectively by

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}$$

and

$$\bigcap_{i \in I} A_i = \{x : x \in A_i \text{ for every } i \in I\}.$$

■ **Example 1.1.1** The following examples illustrate the notation.

(a) Let the index set be $I = \mathbb{N}$ and for each $n \in \mathbb{N}$ we have $A_n = [-n, n]$. Then

$$\bigcup_{n \in \mathbb{N}} A_n = \mathbb{R} \qquad \bigcap_{n \in \mathbb{N}} A_n = [-1, 1].$$

(b) Here we let the index set be $J = (0, 1]$ and for each $\alpha \in J$ we have $A_\alpha = (-\alpha, \alpha)$. Then

$$\bigcup_{\alpha \in J} A_\alpha = (-1, 1) \qquad \bigcap_{\alpha \in J} A_\alpha = \{0\}.$$

The proofs of the following properties are similar to those in Theorem 1.1.2. We include the proof of part (a) and leave the rest as an exercise.

Theorem 1.1.3 Let $\{A_i : i \in I\}$ be an indexed family of subsets of a universal set X and let B be a subset of X. Then the following hold:

(a) $B \cup \left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} B \cup A_i$;
(b) $B \cap \left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} B \cap A_i$;
(c) $B \setminus \left(\bigcap_{i\in I} A_i\right) = \bigcup_{i\in I} B \setminus A_i$;
(d) $B \setminus \left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I} B \setminus A_i$;
(e) $\left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A^c$;
(f) $\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A^c$.

Proof of (a): Let $x \in B \cup \left(\bigcap_{i\in I} A_i\right)$. Then $x \in B$ or $x \in \bigcap_{i\in I} A_i$. If $x \in B$, then $x \in B \cup A_i$ for all $i \in I$ and, thus, $x \in \bigcap_{i\in I} B \cup A_i$. If $x \in \bigcap_{i\in I} A_i$, then $x \in A_i$ for all $i \in I$. Therefore, $x \in B \cup A_i$ for all $i \in I$ and, hence, $x \in \bigcap_{i\in I} B \cup A_i$. We have thus showed $B \cup \left(\bigcap_{i\in I} A_i\right) \subset \bigcap_{i\in I} B \cup A_i$.

Now let $x \in \bigcap_{i\in I} B \cup A_i$. Then $x \in B \cup A_i$ for all $i \in I$. If $x \in B$, then $x \in B \cup \left(\bigcap_{i\in I} A_i\right)$. If $x \notin B$, then we must have that $x \in A_i$ for all $i \in I$. Therefore, $x \in \bigcap_{i\in I} A_i$ and, hence, $x \in B \cup \left(\bigcap_{i\in I} A_i\right)$. This proves the other inclusion and, so, the equality. $\square$

We want to consider pairs of objects in which the order matters. Given objects a and b, we will denote by (a,b) the *ordered pair* where a is the first element and b is the second element. The main characteristic of ordered pairs is that $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$. Thus, the ordered pair $(0,1)$ represents a different object than the pair $(1,0)$ (while the set $\{0,1\}$ is the same as the set $\{1,0\}$)[1].

Given two sets A and B, the *Cartesian product* of A and B is the set defined by

$$A \times B := \{(a,b) : a \in A \text{ and } b \in B\}.$$

■ **Example 1.1.2** If $A = \{1,2\}$ and $B = \{-2,0,1\}$, then

$$A \times B = \{(1,-2),(1,0),(1,1),(2,-2),(2,0),(2,1)\}.$$

■ **Example 1.1.3** If A and B are the intervals $[-1,2]$ and $[0,7]$ respectively, then $A \times B$ is the rectangle

$$[-1,2] \times [0,7] = \{(x,y) : -1 \le x \le 2,\ 0 \le y \le 7\}.$$

We will make use of cartesian products in the next section when we discuss functions.

Exercises

1.1.1 Prove the remaining items in Theorem 1.1.2.

1.1.2 ▶ Let Y and Z be subsets of X. Prove that

$$(X \setminus Y) \cap Z = Z \setminus (Y \cap Z).$$

[1]For a precise definition of ordered pair in terms of sets see [Lay13]

1.1.3 Prove the remaining items in Theorem 1.1.3.

1.1.4 Let A, B, C, and D be sets. Prove the following.

(a) $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

(b) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

(c) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

1.1.5 Let $A \subset X$ and $B \subset Y$. Determine if the following equalities are true and justify your answer:

(a) $(X \times Y) \setminus (A \times B) = (X \setminus A) \times (Y \setminus B)$.

(b) $(X \times Y) \setminus (A \times B) = [(X \setminus A) \times Y] \cup [X \times (Y \setminus B)]$.

1.2 FUNCTIONS

Definition 1.2.1 Let X and Y be sets. A *function from X into Y* is a subset $f \subset X \times Y$ with the following properties

(a) For all $x \in X$ there is $y \in Y$ such that $(x, y) \in f$.

(b) If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

The set X is called the *domain* of f, the set Y is called the *codomain* of f, and we write $f : X \to Y$. The *range* of f is the subset of Y defined by $\{y \in Y : \text{there is } x \in X \text{ such that } (x, y) \in f\}$.

It follows from the definition that, for each $x \in X$, there is exactly one element $y \in Y$ such that $(x, y) \in f$. We will write $y = f(x)$. If $x \in X$, the element $f(x)$ is called the *value of f at x* or the *image of x under f*.

Note that, in this definition, a function is a collection of ordered pairs and, thus, corresponds to the geometric interpretation of the graph of a function given in calculus. In fact, we will refer indistinctly to the function f or to the graph of f. Both refer to the set $\{(x, f(x)) : x \in X\}$.

Let $f : X \to Y$ and $g : X \to Y$ be two functions. Then the two functions are *equal* if they are equal as subsets of $X \times Y$. It is easy to see that f equals g if and only if

$$f(x) = g(x) \text{ for all } x \in X.$$

It follows from the definition that two equal functions must have the same domain.

Let $f : X \to Y$ be a function and let A be a subset of X. The *restriction of f on A*, denoted by $f_{|A}$, is a new function from A into Y given by

$$f_{|A}(a) = f(a) \text{ for all } a \in A.$$

Definition 1.2.2 A function $f : X \to Y$ is called *surjective* (or is said to map X *onto* Y) if for every element $y \in Y$, there exists an element $x \in X$ such that $f(x) = y$.

The function f is called *injective* (or *one-to-one*) if for each pair of distinct elements of X, their images under f are also distinct. Thus, f is one-to-one if and only if for all x and x' in X, the following implication holds:

$$[f(x) = f(x')] \Rightarrow [x = x'].$$

If f is both surjective and injective, it is called *bijective* or a *one-to-one correspondence*. In this case, for any $y \in Y$, there exists a unique element $x \in X$ such that $f(x) = y$. This element x is then denoted by $f^{-1}(y)$. In this way, we already built a function from Y to X called the *inverse* of f.

Theorem 1.2.1 Let $f\colon X \to Y$. If there are two functions $g\colon Y \to X$ and $h\colon Y \to X$ such that $g(f(x)) = x$ for every $x \in X$ and $f(h(y)) = y$ for every $y \in Y$, then f is bijective and $g = h = f^{-1}$.

Proof: First we prove that f is surjective. Let $y \in Y$ and set $x = h(y)$. Then, from the assumption on h, we have $f(x) = f(h(y)) = y$. This shows that f is surjective.

Next we prove that f is injective. Let $x, x' \in X$ be such that $f(x) = f(x')$. Then $x = g(f(x)) = g(f(x')) = x'$. Thus, f is injective.

We have shown that for each $y \in Y$, there is a unique $x \in X$, which we denote $f^{-1}(y)$ such that $f(x) = y$. Since for such a y, $g(y) = g(f(x)) = x$, we obtain $g(y) = f^{-1}(y)$. Since $f(h(y)) = y$, we also conclude that $h(y) = x = f^{-1}(y)$. $\square$

■ **Example 1.2.1** Consider the function $f\colon (1,2] \to [3,4)$ given by $f(x) = 4 - (x-1)^2$. We show that f is bijective. First let $x, y \in (1,2]$ be such that $f(x) = f(y)$. That is, $4 - (x-1)^2 = 4 - (y-1)^2$. Then $(x-1)^2 = (y-1)^2$. Since both $x > 1$ and $y > 1$, we conclude that $x - 1 = y - 1$ and, so, $x = y$. This proves f is injective.

Next let $y \in [3,4)$. We want $x \in (1,2]$ such that $f(x) = y$. Let us set up $4 - (x-1)^2 = y$ and solve for x. We get, $x = \sqrt{4-y} + 1$. Note that since $y < 4$, $y - 4$ has a square root. Also note that since $3 \le y < 4$, we have $1 \ge 4 - y > 0$ and, hence, $2 \ge \sqrt{4-y} + 1 > 1$. Therefore, $x \in (1,2]$. This proves f is surjective.

Definition 1.2.3 Let $f\colon X \to Y$ be a function and let A be a subset of X. Then the *image of A under f* is given by

$$f(A) = \{f(a) : a \in A\}.$$

It follows from the definition that

$$f(A) = \{b \in Y : b = f(a) \text{ for some } a \in A\}.$$

Moreover, f is surjective if and only if $f(X) = Y$.

For a subset B of Y, the *preimage of B under f* is defined by

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

Remark 1.2.2 Note that, despite the notation, the definition of preimage does not require the function to have an inverse. It does not even require the function to be injective. The examples below illustrate these concepts.

■ **Example 1.2.2** Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3x - 1$. Let $A = [0,2)$ and $B = \{1, -4, 5\}$. Then $f(A) = [-1,5)$ and $f^{-1}(B) = \{\frac{2}{3}, -1, 2\}$.

■ **Example 1.2.3** Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = -x + 7$. Let $A = [0,2)$ and $B = (-\infty, 3]$. Then $f(A) = (5,7]$ and $f^{-1}(B) = [4, \infty)$.

■ **Example 1.2.4** Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Let $A = (-1,2)$ and $B = [1,4)$. Then $f(A) = [0,4)$ and $f^{-1}(B) = (-2,-1] \cup [1,2)$.

Theorem 1.2.3 Let $f\colon X \to Y$ be a function, let A be a subset of X, and let B be a subset of Y. The following hold:

(a) $A \subset f^{-1}(f(A))$.

(b) $f(f^{-1}(B)) \subset B$.

Proof: We prove (a) and leave (b) as an exercise.

(a) Let $x \in A$. By the definition of image, $f(x) \in f(A)$. Now, by the definition of preimage, $x \in f^{-1}(f(A))$. $\square$

Theorem 1.2.4 Let $f\colon X \to Y$ be a function, let $A, B \subset X$, and let $C, D \subset Y$. The following hold:

(a) If $C \subset D$, then $f^{-1}(C) \subset f^{-1}(D)$;
(b) $f^{-1}(D \setminus C) = f^{-1}(D) \setminus f^{-1}(C)$;
(c) If $A \subset B$, then $f(A) \subset f(B)$;
(d) $f(A \setminus B) \supset f(A) \setminus f(B)$.

Proof: We prove (b) and leave the other parts as an exercise.

(b) We prove first that $f^{-1}(D \setminus C) \subset f^{-1}(D) \setminus f^{-1}(C)$. Let $x \in f^{-1}(D \setminus C)$. Then, from the definition of inverse image, we get $f(x) \in D \setminus C$. Thus, $f(x) \in D$ and $f(x) \notin C$. Hence $x \in f^{-1}(D)$ and $x \notin f^{-1}(C)$. We conclude that $x \in f^{-1}(D) \setminus f^{-1}(C)$.

Next we prove $f^{-1}(D) \setminus f^{-1}(C) \subset f^{-1}(D \setminus C)$. Let $x \in f^{-1}(D) \setminus f^{-1}(C)$. Thus, $x \in f^{-1}(D)$ and $x \notin f^{-1}(C)$. Therefore, $f(x) \in D$ and $f(x) \notin C$. This means $f(x) \in D \setminus C$ and, so, $x \in f^{-1}(D \setminus C)$. $\square$

Theorem 1.2.5 Let $f\colon X \to Y$ be a function, let $\{A_\alpha\}_{\alpha \in I}$ be an indexed family of subsets of X, and let $\{B_\beta\}_{\beta \in J}$ be an indexed family of subsets of Y. The following hold:

(a) $f(\bigcup_{\alpha \in I} A_\alpha) = \bigcup_{\alpha \in I} f(A_\alpha)$;
(b) $f(\bigcap_{\alpha \in I} A_\alpha) \subset \bigcap_{\alpha \in I} f(A_\alpha)$;
(c) $f^{-1}(\bigcup_{\beta \in J} B_\beta) = \bigcup_{\beta \in J} f^{-1}(B_\beta)$;
(d) $f^{-1}(\bigcap_{\beta \in J} B_\beta) = \bigcap_{\beta \in J} f^{-1}(B_\beta)$.

Proof: We prove (a) and leave the other parts as an exercise.

(a) Let $y \in f(\bigcup_{\alpha \in I} A_\alpha)$. From the definition of image of a set, there is $x \in \bigcup_{\alpha \in I} A_\alpha$ such that $y = f(x)$. From the definition of union of a family of sets, there is $\alpha_0 \in I$ such that $x \in A_{\alpha_0}$. Therefore, $y = f(x) \in f(A_{\alpha_0})$ and, so, $y \in \bigcup_{\alpha \in I} f(A_\alpha)$. $\square$.

Given functions $f\colon X \to Y$ and $g\colon Y \to Z$, we define the composition function $g \circ f$ of f and g as the function $g \circ f\colon X \to Z$ given by

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in X.$$

Theorem 1.2.6 Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two functions and let $B \subset Z$. The following hold:

(a) $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$;
(b) If f and g are injective, then $g \circ f$ is injective;
(c) If f and g are surjective, then $g \circ f$ is surjective;
(d) If $g \circ f$ is injective, then f is injective;
(e) If $g \circ f$ is surjective, then g is surjective.

Proof: We prove (d) and leave the other parts as an exercise.

(d) Suppose $g \circ f$ is injective and let $x, x' \in X$ be such that $f(x) = f(x')$. Then $(g \circ f)(x) = g(f(x)) = g(f(x')) = (g \circ f)(x')$. Since $g \circ f$ is injective, it follows that $x = x'$. We conclude that f is injective. □

Definition 1.2.4 A *sequence* of elements of a set A is a function with domain $\mathbb{N}$ and codomain A.

We discuss sequences in detail in Chapter 2.

Definition 1.2.5 We say that set A is *finite* if it is empty or if there exists a natural number n and a one-to-one correspondence $f\colon A \to \{1, 2, \ldots, n\}$. A set is *infinite* if it is not finite.

We leave it as an exercise to prove that the union of two finite sets is finite. It is also easy to show, by contradiction, that $\mathbb{N}$ is infinite. The following result will be useful when studying sequences and accumulation points.

Theorem 1.2.7 Suppose A is an infinite set. Then there exists a one-to-one function $f\colon \mathbb{N} \to A$.

Proof: Let A be an infinite set. We define f as follows. Choose any element $a_1 \in A$ and set $f(1) = a_1$. Now the set $A \setminus \{a_1\}$ is again infinite (otherwise $A = \{a\} \cup (A \setminus \{a_1\})$ would be the union of two finite sets). So we may choose $a_2 \in A$ with $a_2 \neq a_1$ and we define $f(2) = a_2$[2]. Having defined $f(1), \ldots, f(k)$, we choose $a_{k+1} \in A$ such that $a_{k+1} \in A \setminus \{a_1, \ldots, a_k\}$ and define $f(k+1) = a_{k+1}$ (such an a_{k+1} exists because $A \setminus \{a_1, \ldots, a_k\}$ is infinite and, so, nonempty). The function f so defined clearly has the desired properties. □

To paraphrase, the previous theorem says that in every infinite set we can find a sequence made up of all distinct points.

Exercises

1.2.1 ▶ Let $f\colon X \to Y$ be a function. Prove that:

(a) If f is one-to-one, then $A = f^{-1}(f(A))$ for every subset A of X.
(b) If f is onto, then $f(f^{-1}(B)) = B$ for every subset B of Y.

1.2.2 Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 - 3$ and let $A = [-2, 1)$ and $B = (-1, 6)$. Find $f(A)$ and $f^{-1}(B)$.

1.2.3 Prove that each of the following functions is bijective.

(a) $f\colon (-\infty, 3] \to [-2, \infty)$ given by $f(x) = |x - 3| - 2$.
(b) $g\colon (1, 2) \to (3, \infty)$ given by $g(x) = \dfrac{3}{x - 1}$.

1.2.4 Prove that if $f\colon X \to Y$ is injective, then the following hold:

(a) $f(A \cap B) = f(A) \cap f(B)$ for $A, B \subset X$.
(b) $f(A \setminus B) = f(A) \setminus f(B)$ for $A, B \subset X$.

1.2.5 Prove part (2) of Theorem 1.2.3.

[2] This fact relies on a basic axiom of set theory called the Axiom of Choice. See [Lay13] for more details.

1.2.6 Prove parts (1), (3), and (4) of Theorem 1.2.4.

1.2.7 Prove parts (2), (3), and (4) of Theorem 1.2.5.

1.2.8 Prove parts (1), (2), (3), and (5) of Theorem 1.2.6.

1.2.9 Prove that the union of two finite sets is finite. *Hint:* it is easier to show when the sets are disjoint.

1.3 THE NATURAL NUMBERS AND MATHEMATICAL INDUCTION

We will assume familiarity with the set $\mathbb{N}$ of natural numbers, with the usual arithmetic operations of addition and multiplication on $\mathbb{N}$, and with the notion of what it means for one natural number to be less than another.

In addition, we will also assume the following property of the natural numbers.

Well-Ordering Property of the Natural Numbers: If A is a nonempty subset of $\mathbb{N}$, then there exists an element $\ell \in A$ such that $\ell \leq x$ for all $x \in A$.

To paraphrase the previous property, every nonempty subset of positive integers has a smallest element.

The principle of mathematical induction is a useful tool for proving facts about sequences.

Theorem 1.3.1 — Principle of Mathematical Induction. For each natural number $n \in \mathbb{N}$, suppose that $P(n)$ denotes a proposition which is either true or false. Let $A = \{n \in \mathbb{N} : P(n) \text{ is true}\}$. Suppose the following conditions hold:

(a) $1 \in A$.
(b) For each $k \in \mathbb{N}$, if $k \in A$, then $k+1 \in A$.

Then $A = \mathbb{N}$.

Proof: Suppose conditions (a) and (b) hold. Assume, by way of contradiction, that $A \neq \mathbb{N}$. Set $B = \mathbb{N} \setminus A$, that is, $B = \{n \in \mathbb{N} : P(n) \text{ is false}\}$. Then B is a nonempty subset of $\mathbb{N}$. By the Well-Ordering Property of the natural numbers, there exists a smallest element $\ell \in B$. By condition (a), $1 \notin B$. Hence, $\ell \geq 2$. It follows that $k = \ell - 1$ is a natural number. Since $k < \ell$, $k \notin B$ and, hence, we have that $P(k)$ is true. By condition (b), we obtain that $P(k+1)$ is true. But $k+1 = \ell$, and $P(\ell)$ is false, since $\ell \in B$. This is a contradiction, so the conclusion follows. $\square$

To paraphrase, the principle says that, given a list of propositions $P(n)$, one for each $n \in \mathbb{N}$, if $P(1)$ is true and, moreover, $P(k+1)$ is true whenever $P(k)$ is true, then all propositions are true.

We will refer to this principle as mathematical induction or simply induction. Condition (a) above is called the *base case* and condition (b) the *inductive step*. When proving (b), the statement $P(k)$ is called the *inductive hypothesis*.

■ **Example 1.3.1** Prove using induction that for all $n \in \mathbb{N}$

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

The statement $P(n)$ is the equality $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. Now the base case says that $1 = \frac{1(1+1)}{2}$, which is clearly true.

Suppose $P(k)$ is true for some $k \in \mathbb{N}$. That is, suppose that $1+2+\cdots+k = \frac{k(k+1)}{2}$ (this is the inductive hypothesis). Now we have

$$1+2+\cdots+k+(k+1) = \frac{k(k+1)}{2}+(k+1) = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

This shows that $P(k+1)$ is true. We have now proved conditions (a) and (b) of Theorem 1.3.1. Therefore, by the principle of mathematical induction we conclude that

$$1+2+\cdots+n = \frac{n(n+1)}{2} \text{ for all } n \in \mathbb{N}.$$

■ **Example 1.3.2** Prove using induction that for all $n \in \mathbb{N}$, $7^n - 2^n$ is divisible by 5.

For $n = 1$, we have $7-2 = 5$, which is clearly a multiple of 5.

Suppose that $7^k - 2^k$ is a multiple of 5 for some $k \in \mathbb{N}$. That is, there is an integer j such that $7^k - 2^k = 5j$. Let us write $7^k = 2^k + 5j$. Now, substituting this expression below, we have

$$7^{k+1}-2^{k+1} = 7\cdot 7^k - 2\cdot 2^k = 7(2^k+5j)-2\cdot 2^k = 7\cdot 2^k - 2\cdot 2^k + 7\cdot 5j = 2^k(7-2)+5\cdot 7j = 5(2^k+7j).$$

It follows that $7^{k+1}-2^{k+1}$ is a multiple of 5. This proves the inductive step.

We conclude by induction that $7^n - 2^n$ is divisible by 5 for all $n \in \mathbb{N}$.

■ **Example 1.3.3** Prove using induction that for all $n \in \mathbb{N}$

$$n+1 \leq 2^n$$

For $n = 1$, we have $1+1 = 2 = 2^1$, so the base case is true.

Suppose next that $k+1 \leq 2^k$ for some $k \in \mathbb{N}$. Then $k+1+1 \leq 2^k+1$. Since 2^k is a positive integer, we also have $1 \leq 2^k$. Therefore,

$$(k+1)+1 \leq 2^k+1 \leq 2^k+2^k = 2\cdot 2^k = 2^{k+1}.$$

We conclude by the principle of mathematical induction that $n+1 \leq 2^n$ for all $n \in \mathbb{N}$.

The following result is known as the *Generalized Principle of Mathematical Induction*. It simply states that we can start the induction process at any integer n_0, and then we obtain the truth of all statements $P(n)$ for $n \geq n_0$.

Theorem 1.3.2 — Generalized Principle of Mathematical Induction. Let $n_0 \in \mathbb{N}$ and for each natural $n \geq n_0$, suppose that $P(n)$ denotes a proposition which is either true or false. Let $A = \{n \in \mathbb{N} : P(n) \text{ is true}\}$. Suppose the following two conditions hold:

(a) $n_0 \in A$.

(b) For each $k \in \mathbb{N}$, $k \geq n_0$, if $k \in A$, then $k+1 \in A$.

Then $\{k \in \mathbb{N} : k \geq n_0\} \subset A$.

Proof: Suppose conditions (a) and (b) hold. Assume, by way of contradiction, that $A \not\supseteq \{k \in \mathbb{N} : k \geq n_0\}$. Set $B = \{n \in \mathbb{N} : n \geq n_0, P(n) \text{ is false}\}$. Then B is a nonempty subset of $\mathbb{N}$. By the Well-Ordering Property of the natural numbers, there exists a smallest element $\ell \in B$. By condition (a), $n_0 \notin B$. Hence, $\ell \geq n_0+1$. It follows that $k = \ell - 1 \geq n_0$. Since $k < \ell$, $k \notin B$ and, so, we have that $P(k)$ is true. By condition (b), we obtain that $P(k+1)$ is true. But $k+1 = \ell$, and $P(\ell)$ is false, since $\ell \in B$. This is a contradiction, so the conclusion follows. □

■ **Example 1.3.4** Prove by induction that $3n < 2^n$ for all $n \geq 4$.

The statement is true for $n = 4$ since $12 < 16$. Suppose next that $3k < 2^k$ for some $k \in \mathbb{N}$, $k \geq 4$. Now,

$$3(k+1) = 3k + 3 < 2^k + 3 < 2^k + 2^k = 2^{k+1}, \tag{1.1}$$

where the second inequality follows since $k \geq 4$ and, so, $2^k \geq 16 > 3$. This shows that $P(k+1)$ is true. Thus, by the generalized principle of induction, the inequality holds for all $n \geq 4$.

Next we present another variant of the induction principle which makes it easier to prove the inductive step. Despite its name, this principle is equivalent to the standard one.

Theorem 1.3.3 — Principle of Strong Induction. For each natural $n \in \mathbb{N}$, suppose that $P(n)$ denotes a proposition which is either true or false. Let $A = \{n \in \mathbb{N} : P(n) \text{ is true}\}$. Suppose the following two conditions hold:

(a) $1 \in A$.

(b) For each $k \in \mathbb{N}$, if $1, 2, \ldots, k \in A$, then $k + 1 \in A$.

Then $A = \mathbb{N}$.

Remark 1.3.4 Note that the inductive step above says that, in order to prove $P(k+1)$ is true, we may assume not only that $P(k)$ is true, but also that $P(1)$, $P(2)$,...,$P(k-1)$ are true.

There is also a generalized version of this theorem where the base case is for some integer $n_0 > 1$.

■ **Example 1.3.5** Prove by induction that every positive integer greater than 1 is either a prime number or a product of prime numbers.

Clearly, the statement is true for $n = 2$. Suppose the statement holds for any positive integer $m \in \{2, \ldots, k\}$, where $k \in \mathbb{N}$, $k \geq 2$. If $k + 1$ is prime, the statement holds for $k + 1$. Otherwise, there are positive integers $p, q > 1$ such that $k + 1 = pq$. Since $p, q \leq k$, by the inductive assumption applied to both p and q we can find prime numbers $r_1, \ldots, r_\ell$ and $s_1, \ldots, s_m$ such that $p = r_1 \cdots r_\ell$ and $q = s_1 \cdots s_m$ (note that ℓ and m may both equal 1). But then

$$k + 1 = r_1 \cdots r_\ell s_1 \cdots s_m. \tag{1.2}$$

Thus, the statement holds true for $k + 1$. The conclusion now follows by the Principle of Strong Induction.

Exercises

1.3.1 Prove the following using Mathematical Induction.

(a) $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.

(b) $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for all $n \in \mathbb{N}$.

(c) $1 + 3 + \cdots + (2n-1) = n^2$ for all $n \in \mathbb{N}$.

1.3.2 Prove that for all $n \in \mathbb{N}$, $9^n - 5^n$ is divisible by 4.

1.3.3 Prove that for all $n \in \mathbb{N}$, $7^n - 1$ is divisible by 3.

1.3.4 Prove the following using induction.

(a) $2n+1 \leq 2^n$ for $n \geq 3$ ($n \in \mathbb{N}$).
(b) $n^2 \leq 3^n$ for all $n \in \mathbb{N}$. (*Hint:* show first that for all $n \in \mathbb{N}$, $2n \leq n^2+1$. This does not require induction.)
(c) $n^3 \leq 3^n$ for all $n \in \mathbb{N}$. (*Hint:* Check the cases $n=1$ and $n=2$ directly and then use induction for $n \geq 3$.)

1.3.5 Given a real number $a \neq 1$, prove that

$$1+a+a^2+\cdots+a^n = \frac{1-a^{n+1}}{1-a} \text{ for all } n \in \mathbb{N}.$$

1.3.6 ▶ The Fibonacci sequence is defined by

$$a_1 = a_2 = 1 \qquad \text{and} \qquad a_{n+2} = a_{n+1} + a_n \text{ for } n \geq 1.$$

Prove that

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

1.3.7 Let $a \geq -1$. Prove by induction that

$$(1+a)^n \geq 1+na \text{ for all } n \in \mathbb{N}.$$

1.3.8 ▷ Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$. Use Mathematical Induction to prove the binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k},$$

where $\displaystyle\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

1.4 ORDERED FIELD AXIOMS

In this book, we will start from an axiomatic presentation of the real numbers. That is, we will assume that there exists a set, denoted by $\mathbb{R}$, satisfying the ordered field axioms, stated below, together with the completeness axiom, presented in the next section. In this way we identify the basic properties that characterize the real numbers. After listing the ordered field axioms we derive from them additional familiar properties of the real numbers. We conclude the section with the definition of absolute value of a real number and with several results about it that will be used often later in the text.

We assume the existence of a set $\mathbb{R}$ (the set of real numbers) and two operations $+$ and $\cdot$ (addition and multiplication) assigning to each pair of real numbers x, y, unique real numbers $x+y$ and $x \cdot y$ and satisfying the following properties:

(1a) $(x+y)+z = x+(y+z)$ for all $x, y, z \in \mathbb{R}$.
(1b) $x+y = y+x$ for all $x, y \in \mathbb{R}$.
(1c) There exists a unique element $0 \in \mathbb{R}$ such that $x+0 = x$ for all $x \in \mathbb{R}$.

(1d) For each $x \in \mathbb{R}$, there exists a unique element $-x \in \mathbb{R}$ such that $x + (-x) = 0$.

(2a) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in \mathbb{R}$.
(2b) $x \cdot y = y \cdot x$ for all $x, y \in \mathbb{R}$.
(2c) There exists a unique element $1 \in \mathbb{R}$ such that $1 \neq 0$ and $x \cdot 1 = x$ for all $x \in \mathbb{R}$.
(2d) For each $x \in \mathbb{R} \setminus \{0\}$, there exists a unique element $x^{-1} \in \mathbb{R}$ such that $x \cdot (x^{-1}) = 1$. (We also write $1/x$ instead of x^{-1}.)
(2e) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in \mathbb{R}$.

We often write xy instead of $x \cdot y$.

In addition to the *algebraic axioms* above, there is a relation $<$ on $\mathbb{R}$ that satisfies the *order axioms* below:

(3a) For all $x, y \in \mathbb{R}$, exactly one of the three relations holds: $x = y$, $y < x$, or $x < y$.
(3b) For all $x, y, z \in \mathbb{R}$, if $x < y$ and $y < z$, then $x < z$.
(3c) For all $x, y, z \in \mathbb{R}$, if $x < y$, then $x + z < y + z$.
(3d) For all $x, y, z \in \mathbb{R}$, if $x < y$ and $0 < z$, then $xz < yz$.

We will use the notation $x \leq y$ to mean $x < y$ or $x = y$. We may also use the notation $x > y$ to represent $y < x$ and the notation $x \geq y$ to mean $x > y$ or $x = y$.

A set $\mathbb{F}$ together with two operations $+$ and $\cdot$ and a relation $<$ satifying the 13 axioms above is called an *ordered field*. Thus, the real numbers are an example of an ordered field. Another example of an ordered field is the set of rational numbers $\mathbb{Q}$ with the familiar operations and order. The integers $\mathbb{Z}$ do not form a field since for an integer m other than 1 or -1, its reciprocal $1/m$ is not an integer and, thus, axiom 2(d) above does not hold. In particular, the set of positive integers $\mathbb{N}$ does not form a field either. As mentioned above the real numbers $\mathbb{R}$ will be defined as the ordered field which satisfies one additional property described in the next section: the completeness axiom.

From these axioms, many familiar properties of $\mathbb{R}$ can be derived. Some examples are given in the next proposition. The proof illustrates how the given axioms are used at each step of the derivation.

Proposition 1.4.1 For $x, y, z \in \mathbb{R}$, the following hold:

(a) If $x + y = x + z$, then $y = z$;
(b) $-(-x) = x$;
(c) If $x \neq 0$ and $xy = xz$, then $y = z$;
(d) If $x \neq 0$, then $1/(1/x) = x$;
(e) $0x = 0 = x0$;
(f) $-x = (-1)x$;
(g) $x(-z) = (-x)z = -(xz)$.
(h) If $x > 0$, then $-x < 0$; if $x < 0$, then $-x > 0$;
(i) If $x < y$ and $z < 0$, then $xz > yz$;
(j) $0 < 1$.

Proof: (a) Suppose $x+y=x+z$. Adding $-x$ (which exists by axiom (2d)) to both sides, we have

$$(-x)+(x+y)=(-x)+(x+z).$$

Then axiom (1a) gives

$$[(-x)+x]+y=[(-x)+x]+z.$$

Thus, again by axiom (2d), $0+y=0+z$ and, by axiom (1c), $y=z$.

(b) Since $(-x)+x=0$, we have (by uniqueness in axiom (2d)) $-(-x)=x$.

The proofs of (c) and (d) are similar.

(e) Using axiom (2e) we have $0x=(0+0)x=0x+0x$. Adding $-(0x)$ to both sides (axiom (2d)) and using axioms (1a) and (1c), we get

$$0=-(0x)+0x=-(0x)+(0x+0x)=(-(0x)+0x)+0x=0+0x=0x.$$

That $0x=x0$ follows from axiom (2b).

(f) Using axioms (2c) and (2e) we get $x+(-1)x=1x+(-1)x=(1+(-1))x$. From axiom (2d) we get $1+(-1)=0$ and from part (e) we get $x+(-1)x=0x=0$. From the uniqueness in axiom (2d) we get $(-1)x=-x$ as desired.

(g) Using axioms (2e) and (1c) we have $xz+x(-z)=x(z+(-z))=x0=0$. Thus, using axiom (2d) we get that $x(-z)=-(xz)$. The other equality follows similarly.

(h) From $x>0$, using axioms (3c) and (1c) we have $x+(-x)>0+(-x)=-x$. Thus, using axiom (2d), we get $0>-x$. The other case follows in a similar way.

(i) Since $z<0$, by part (h), $-z>0$. Then, by axiom (3d), $x(-z)<y(-z)$. Combining this with part (g) we get $-xz<-yz$. Adding $xz+yz$ to both sides and using axioms (1a), (3c), (1b), and (1c) we get

$$xy=(-xz+xz)+xy=-xz+(xz+xy)<-xy+(xz+xy)=-xy+(xy+xz)=(-xy+xy)+xz=xz.$$

(j) Axiom (2c) gives that $1\neq 0$. Suppose, by way of contradiction, that $1<0$. Then by part (i), $1\cdot 1>0\cdot 1$. Since $1\cdot 1=1$, by axiom (2c) and $0\cdot 1=0$ by part (e), we get $1>0$ which is a contradiction. It follows that $1>0$. $\square$

Note that we can assume that the set of all natural numbers is a subset of $\mathbb{R}$ (and of any ordered field, in fact) by identifying the 1 in $\mathbb{N}$ with the 1 in axiom (2c) above, the number 2 with $1+1$, 3 with $1+1+1$, etc. Furthermore, since $0<1$ (from part (j) of the previous proposition), axiom (3c) gives, $1<2<3$, etc (in particular all these numbers are distinct). In a similar way, can include $\mathbb{Z}$ and $\mathbb{Q}$ as subsets.

We say that a real number x is *irrational* if $x\in\mathbb{R}\setminus\mathbb{Q}$, that is, if it is not rational.

Definition 1.4.1 Given $x\in\mathbb{R}$, define the *absolute value* of x by

$$|x|=\begin{cases}x, & \text{if } x\geq 0;\\ -x, & \text{if } x<0.\end{cases}$$

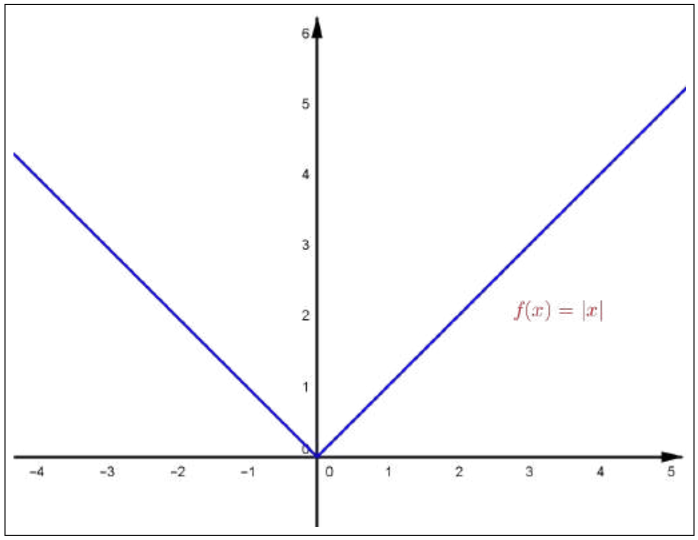

Figure 1.1: The absolute value function.

The following properties of absolute value follow directly from the definition.

Proposition 1.4.2 Let $x, y, M \in \mathbb{R}$ and suppose $M > 0$. The following properties hold:

(a) $|x| \geq 0$;
(b) $|-x| = |x|$;
(c) $|xy| = |x||y|$;
(d) $|x| < M$ if and only if $-M < x < M$. (The same holds if $<$ is replaced with $\leq$.)

Proof: We prove (d) and leave the other parts as an exercise.

(d) Suppose $|x| < M$. In particular, this implies $M > 0$. We consider the two cases separately: $x \geq 0$ and $x < 0$. Suppose first $x \geq 0$. Then $|x| = x$ and, hence, $-M < 0 \leq x = |x| < M$. Now suppose $x < 0$. Then $|x| = -x$. Therefore, $-x < M$ and, so $x > -M$. It follows that $-M < x < 0 < M$.

For the converse, suppose $-M < x < M$. Again, we consider different cases. If $x \geq 0$, then $|x| = x < M$ as desired. Next suppose $x < 0$. Now, $-M < x$ implies $M > -x$. Then $|x| = -x < M$. $\square$

Note that as a consequence of part (d) above, since $|x| \leq |x|$ we get $-|x| \leq x \leq |x|$.

The next theorem will play an important role in the study of limits.

Theorem 1.4.3 — Triangle Inequality. Given $x, y \in \mathbb{R}$,

$$|x+y| \leq |x| + |y|.$$

Proof: From the observation above, we have

$$-|x| \leq x \leq |x|$$

$$-|y| \leq y \leq |y|.$$

Adding up the inequalities gives

$$-|x| - |y| \leq x + y \leq |x| + |y|.$$

Since $-|x|-|y| = -(|x|+|y|)$, the conclusion follows from Proposition 1.4.2 (d). □

Corollary 1.4.4 For any $x, y \in \mathbb{R}$,

$$||x|-|y|| \leq |x-y|.$$

Remark 1.4.5 The absolute value has a geometric interpretation when considering the numbers in an ordered field as points on a line. The number $|a|$ denotes the distance from the number a to 0. More generally, the number $d(a,b) = |a-b|$ is the distance between the points a and b. It follows easily from Proposition 1.4.2 that $d(x,y) \geq 0$, and $d(x,y) = 0$ if and only if $x = y$. Moreover, the triangle inequality implies that

$$d(x,y) \leq d(x,z) + d(z,y),$$

for all numbers x, y, z.

Exercises

1.4.1 Prove that n is an even integer if and only if n^2 is an even integer. (*Hint:* prove the "if" part by contraposition, that is, prove that if n is odd, then n^2 is odd.)

1.4.2 Prove parts (c) and (d) of Proposition 1.4.1

1.4.3 Let $a, b, c, d \in \mathbb{R}$. Suppose $0 < a < b$ and $0 < c < d$. Prove that $ac < bd$.

1.4.4 Prove parts (a), (b), and (c) of Proposition 1.4.2.

1.4.5 ▶ Prove Corollary 1.4.4.

1.4.6 Given two real numbers x and y, prove that

$$\max\{x,y\} = \frac{x+y+|x-y|}{2} \text{ and } \min\{x,y\} = \frac{x+y-|x-y|}{2}.$$

1.4.7 Let $x, y, M \in \mathbb{R}$. Prove the following

(a) $|x|^2 = x^2$.
(b) $|x| < M$ if and only if $x < M$ and $-x < M$.
(c) $|x+y| = |x|+|y|$ if and only if $xy \geq 0$.

1.5 THE COMPLETENESS AXIOM FOR THE REAL NUMBERS

There are many examples of ordered fields. However, we are interested in the field of real numbers. There is an additional axiom that will distinguish this ordered field from all others. In order to introduce our last axiom for the real numbers, we first need some definitions.

Definition 1.5.1 Let A be a subset of $\mathbb{R}$. A number M is called an *upper bound* of A if

$$x \leq M \text{ for all } x \in A.$$

If A has an upper bound, then A is said to be *bounded above*.

Similarly, a number L is a *lower bound* of A if

$L \leq x$ for all $x \in A$,

and A is said to be *bounded below* if it has a lower bound. We also say that A is *bounded* if it is both bounded above and bounded below.

It follows that a set A is bounded if and only if there exist $M \in \mathbb{R}$ such that $|x| \leq M$ for all $x \in A$ (see Exercise 1.5.1).

Definition 1.5.2 Let A be a nonempty set that is bounded above. We call a number α a *least upper bound* or *supremum* of A, if

(1) $x \leq \alpha$ for all $x \in A$ (that is, α is an upper bound of A);
(2) If M is an upper bound of A, then $\alpha \leq M$ (this means α is smallest among all upper bounds).

It is easy to see that if A has a supremum, then it has only one (see Exercise 1.5.2). In this case, we denote such a number by $\sup A$.

■ **Example 1.5.1**

(a) $\sup[0,3) = \sup[0,3] = 3$.

First consider the set $[0,3] = \{x \in \mathbb{R} : 0 \leq x \leq 3\}$. By its very definition we see that for all $x \in [0,3]$, $x \leq 3$. Thus 3 is an upper bound. This verifies condition (1) in the definition of supremum. Next suppose M is an upper bound of $[0,3]$. Since $3 \in [0,3]$, we get $3 \leq M$. This verifies condition (2) in the definition of supremum. It follows that 3 is indeed the supremum of $[0,3]$.

Consider next the set $[0,3) = \{x \in \mathbb{R} : 0 \leq x < 3\}$. It follows as before that 3 is an upper bound of $[0,3)$. Now suppose that M is an upper bound of $[0,3)$ and assume by way of contradiction that $3 > M$. If $0 > M$, then M is not an upper bound of $[0,3)$ as 0 is an element of $[0,3)$. If $0 \leq M$, set $x = \frac{M+3}{2}$. Then $0 \leq x < 3$ and $x > M$, which shows M is not an upper bound of $[0,3)$. Since we get a contradiction in both cases, we conclude that $3 \leq M$ and, hence, 3 is the supremum of $[0,3)$.

(b) $\sup\{3,5,7,8,10\} = 10$.

Clearly 10 is an upper bound of the set. Moreover, any upper bound M must satisfy $10 \leq M$ as 10 is an element of the set. Thus 10 is the supremum.

(c) $\sup\left\{\dfrac{(-1)^n}{n} : n \in \mathbb{N}\right\} = \dfrac{1}{2}$.

Note that if $n \in \mathbb{N}$ is even, then $n \geq 2$ and

$$\frac{(-1)^n}{n} = \frac{1}{n} \leq \frac{1}{2}.$$

If $n \in \mathbb{N}$ is odd, then

$$\frac{(-1)^n}{n} = \frac{-1}{n} < 0 < \frac{1}{2}.$$

This shows that $\frac{1}{2}$ is an upper bound of the set. Since $\frac{1}{2}$ is an element of the set, it follows as in the previous example that $\frac{1}{2}$ is the supremum.

(d) $\sup\{x^2 : -2 < x < 1,\, x \in \mathbb{R}\} = 4$.

Set $A = \{x^2 : -2 < x < 1,\, x \in \mathbb{R}\}$. If $y \in A$, then $y = x^2$ for some x satisfying $-2 < x < 1$ and, hence, $|x| < 2$. Therefore, $y = x^2 = |x|^2 < 4$. Thus, 4 is an upper bound of A. Suppose M is an upper bound of A but $M < 4$. Choose a number $y \in \mathbb{R}$ such that $M < y < 4$ and $0 < y$. Set $x = -\sqrt{y}$. Then $-2 < x < 0 < 1$ and, so, $y = x^2 \in A$. However, $y > M$ which contradicts the fact that M is an upper bound. Thus $4 \leq M$. This proves that $4 = \sup A$.

The following proposition is convenient in working with suprema.

Proposition 1.5.1 Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. Then $\alpha = \sup A$ if and only if

(1') $x \leq \alpha$ for all $x \in A$;

(2') For any $\varepsilon > 0$, there exists $a \in A$ such that $\alpha - \varepsilon < a$.

Proof: Suppose first that $\alpha = \sup A$. Then clearly (1') holds (since this is identical to condition (1) in the definition of supremum). Now let $\varepsilon > 0$. Since $\alpha - \varepsilon < \alpha$, condition (2) in the definition of supremum implies that $\alpha - \varepsilon$ is not an upper bound of A. Therefore, there must exist an element a of A such that $\alpha - \varepsilon < a$ as desired.

Conversely, suppose conditions (1') and (2') hold. Then all we need to show is that condition (2) in the definition of supremum holds. Let M be an upper bound of A and assume, by way of contradiction, that $M < \alpha$. Set $\varepsilon = \alpha - M$. By condition (2) in the statement, there is $a \in A$ such that $a > \alpha - \varepsilon = M$. This contradicts the fact that M is an upper bound. The conclusion now follows. □

The following is an axiom of the real numbers and is called the *completeness axiom*.

The Completeness Axiom. *Every nonempty subset A of $\mathbb{R}$ that is bounded above has a least upper bound. That is,* $\sup A$ *exists and is a real number.*

This axiom distinguishes the real numbers from all other ordered fields and it is crucial in the proofs of the central theorems of analysis.

There is a corresponding definition for the infimum of a set.

Definition 1.5.3 Let A be a nonempty subset of $\mathbb{R}$ that is bounded below. We call a number β a *greatest lower bound* or *infimum* of A, denoted by $\beta = \inf A$, if

(1) $x \geq \beta$ for all $x \in A$ (that is, β is a lower bound of A);
(2) If N is a lower bound of A, then $\beta \geq N$ (this means β is largest among all lower bounds).

Using the completeness axiom, we can prove that if a nonempty set is bounded below, then its infimum exists (see Exercise 1.5.5).

■ **Example 1.5.2**

(a) $\inf(0,3] = \inf[0,3] = 0$.

(b) $\inf\{3,5,7,8,10\} = 3$.

(c) $\inf\left\{\frac{(-1)^n}{n} : n \in \mathbb{N}\right\} = -1$.

(d) $\inf\{1 + \frac{1}{n} : n \in \mathbb{N}\} = 1$.

(e) $\inf\{x^2 : -2 < x < 1, x \in \mathbb{R}\} = 0$.

The following proposition is useful when dealing with infima and its proof is completely analogous to that of Proposition 1.5.1.

Proposition 1.5.2 Let A be a nonempty subset of $\mathbb{R}$ that is bounded below. Then $\beta = \inf A$ if and only if

(1') $x \geq \beta$ for all $x \in A$;

(2') For any $\varepsilon > 0$, there exists $a \in A$ such that $a < \beta + \varepsilon$.

The following is a basic property of suprema. Additional ones are described in the exercises.

Theorem 1.5.3 Let A and B be nonempty sets and $A \subset B$. Suppose B is bounded above. Then $\sup A \leq \sup B$.

Proof: Let M be an upper bound for B, then for $x \in B$, $x \leq M$. In particular, it is also true that $x \leq M$ for $x \in A$. Thus, A is also bounded above. Now, since $\sup B$ is an upper bound for B, it is also an upper bound for A. Then, by the second condition in the definition of supremum, $\sup A \leq \sup B$ as desired. $\square$

It will be convenient for the study of limits of sequences and functions to introduce two additional symbols.

Definition 1.5.4 The *extended real number system* consists of the real field $\mathbb{R}$ and the two symbols ∞ and $-\infty$. We preserve the original order in $\mathbb{R}$ and define

$$-\infty < x < \infty$$

for every $x \in \mathbb{R}$

The extended real number system does not form an ordered field, but it is customary to make the following conventions:

(a) If x is a real number, then

$$x + \infty = \infty, \quad x + (-\infty) = -\infty.$$

(b) If $x > 0$, then $x \cdot \infty = \infty$, $\quad x \cdot (-\infty) = -\infty$.

(c) If $x < 0$, then $x \cdot \infty = -\infty$, $\quad x \cdot (-\infty) = \infty$.

(d) $\infty + \infty = \infty$, $-\infty + (-\infty) = -\infty$, $\infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty$, and $(-\infty) \cdot \infty = \infty \cdot (-\infty) = -\infty$.

We denote the extended real number set by $\overline{\mathbb{R}}$. The expressions $0 \cdot \infty$, $\infty + (-\infty)$, and $(-\infty) + \infty$ are left undefined.

The set $\overline{\mathbb{R}}$ with the above conventions will be convenient to describe results about limits in later chapters.

Definition 1.5.5 If $A \neq \emptyset$ is not bounded above in $\mathbb{R}$, we will write $\sup A = \infty$. If A is not bounded below in $\mathbb{R}$, we will write $\inf A = -\infty$.

With this definition, every nonempty subset of $\mathbb{R}$ has a supremum and an infimum in $\overline{\mathbb{R}}$. To complete the picture we adopt the following conventions for the empty set: $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$.

Exercises

1.5.1 Prove that a subset A of $\mathbb{R}$ is bounded if and only if there is $M \in \mathbb{R}$ such that $|x| \leq M$ for all $x \in A$.

1.5.2 Let A be a nonempty set and suppose α and β satisfy conditions (1) and (2) in Definition 1.5.2 (that is, both are suprema of A). Prove that $\alpha = \beta$.

1.5.3 For each subset of $\mathbb{R}$ below, determine if it is bounded above, bounded below, or both. If it is bounded above (below) find the supremum (infimum). Justify all your conclusions.

(a) $\{1, 5, 17\}$
(b) $[0, 5)$
(c) $\left\{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$
(d) $(-3, \infty)$
(e) $\{x \in \mathbb{R} : x^2 - 3x + 2 = 0\}$
(f) $\{x^2 - 3x + 2 : x \in \mathbb{R}\}$
(g) $\{x \in \mathbb{R} : x^3 - 4x < 0\}$
(h) $\{x \in \mathbb{R} : 1 \leq |x| < 3\}$

1.5.4 ▶ Suppose A and B are nonempty subsets of $\mathbb{R}$ that are bounded above. Define

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

Prove that $A + B$ is bounded above and

$$\sup(A + B) = \sup A + \sup B.$$

1.5.5 Let A be a nonempty subset of $\mathbb{R}$. Define $-A = \{-a : a \in A\}$.

(a) Prove that if A is bounded below, then $-A$ is bounded above.
(b) Prove that if A is bounded below, then A has an infimum in $\mathbb{R}$ and $\inf A = -\sup(-A)$.

1.5.6 Let A be a nonempty subset of $\mathbb{R}$ and $\alpha \in \mathbb{R}$. Define $\alpha A = \{\alpha a : a \in A\}$. Prove the following statements:

(a) If $\alpha > 0$ and A is bounded above, then αA is bounded above and $\sup \alpha A = \alpha \sup A$.
(b) If $\alpha < 0$ and A is bounded above, then αA is bounded below and $\inf \alpha A = \alpha \sup A$.

1.5.7 Suppose A and B are nonempty subsets of $\mathbb{R}$ that are bounded below. Prove that $A + B$ is bounded below and

$$\inf(A + B) = \inf A + \inf B.$$

1.5.8 Let A, B be nonempty subsets of $\mathbb{R}$ that are bounded below. Prove that if $A \subset B$, then

$$\inf A \geq \inf B.$$

1.6 APPLICATIONS OF THE COMPLETENESS AXIOM

We prove here several fundamental properties of the real numbers that are direct consequences of the Completeness Axiom.

Theorem 1.6.1 — The Archimedean Property. The set of natural numbers is unbounded above.

Proof: Let us assume by contradiction that $\mathbb{N}$ is bounded above. Since $\mathbb{N}$ is nonempty,

$$\alpha = \sup \mathbb{N}$$

exists and is a real number. By Proposition 1.5.1 (with $\varepsilon = 1$), there exists $n \in \mathbb{N}$ such that

$$\alpha - 1 < n \leq \alpha.$$

But then $n + 1 > \alpha$. This is a contradiction since $n + 1$ is a natural number. □

The following theorem presents several immediate consequences.

Theorem 1.6.2 The following hold:

(a) For any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x < n$;
(b) For any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $1/n < \varepsilon$;
(c) For any $x > 0$ and for any $y \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $y < nx$;
(d) For any $x \in \mathbb{R}$, there exists $m \in \mathbb{Z}$ such that $m - 1 \leq x < m$.

Proof: (a) Fix any $x \in \mathbb{R}$. Since $\mathbb{N}$ is not bounded above, x cannot be an upper bound of $\mathbb{N}$. Thus, there exists $n \in \mathbb{N}$ such that $x < n$.

(b) Fix any $\varepsilon > 0$. Then $1/\varepsilon$ is a real number. By (1), there exists $n \in \mathbb{N}$ such that

$$1/\varepsilon < n.$$

This implies $1/n < \varepsilon$.

(c) We only need to apply (a) for the real number y/x.

(d) First we consider the case where $x > 0$. Define the set

$$A = \{n \in \mathbb{N} : x < n\}.$$

From part (a), A is nonempty. Since A is a subset of $\mathbb{N}$, by the Well-Ordering Property of the natural numbers, A has a smallest element ℓ. In particular, $x < \ell$ and $\ell - 1$ is not in A. Since $\ell \in \mathbb{N}$, either $\ell - 1 \in \mathbb{N}$ or $\ell - 1 = 0$. If $\ell - 1 \in \mathbb{N}$, since $\ell - 1 \notin A$ we get $\ell - 1 \leq x$. If $\ell - 1 = 0$, we have $\ell - 1 = 0 < x$. Therefore, in both cases we have $\ell - 1 \leq x < \ell$ and the conclusion follows with $m = \ell$.

In the case $x \leq 0$, by part (1), there exists $N \in \mathbb{N}$ such that

$$|x| < N.$$

In this case, $-N < x < N$, so $x + N > 0$. Then, by the result just obtained for positive numbers, there exists a natural number k such that $k - 1 \leq x + N < k$. This implies

$$k - N - 1 \leq x < k - N.$$

Setting $m = k - N$, the conclusion follows. The proof is now complete. □

■ **Example 1.6.1** Let $A = \sup\{1 - \frac{1}{n} : n \in \mathbb{N}\}$. We claim that $\sup A = 1$.

We use Proposition 1.5.1. Since $1 - 1/n < 1$ for all $n \in \mathbb{N}$, we obtain condition (1'). Next, let $\varepsilon > 0$. From Theorem 1.6.2 (b) we can find $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. Then

$$1 - \varepsilon < 1 - \frac{1}{n}.$$

This proves condition (2') with $a = 1 - \frac{1}{n}$ and completes the proof.

Theorem 1.6.3 — The Density Property of $\mathbb{Q}$. If x and y are two real numbers such that $x < y$, then there exists a rational number r such that

$$x < r < y.$$

Proof: We are going to prove that there exist an integer m and a positive integer n such that

$$x < m/n < y,$$

or, equivalently,

$$nx < m < ny = nx + n(y - x).$$

Since $y - x > 0$, by Theorem 1.6.2 (3), there exists $n \in \mathbb{N}$ such that $1 < n(y - x)$. Then

$$ny = nx + n(y - x) > nx + 1.$$

By Theorem 1.6.2 (4), one can choose $m \in \mathbb{Z}$ such that

$$m - 1 \le nx < m.$$

Then $nx < m \le nx + 1 < ny$. Therefore,

$$x < m/n < y.$$

The proof is now complete. □

We will prove in a later section (see Examples 3.4.2 and 4.3.1) that there exists a (unique) positive real number x such that $x^2 = 2$. We denote that number by $\sqrt{2}$. The following result shows, in particular, that $\mathbb{R} \neq \mathbb{Q}$.

Proposition 1.6.4 The number $\sqrt{2}$ is irrational.

Proof: Suppose, by way of contradiction, that $\sqrt{2} \in \mathbb{Q}$. Then there are integers r and s with $s \neq 0$, such that

$$\sqrt{2} = \frac{r}{s}.$$

By canceling out the common factors of r and s, we may assume that r and s have no common factors.

Now, by squaring both sides of the equation above, we get

$$2 = \frac{r^2}{s^2},$$

and, hence,

$$2s^2 = r^2. \tag{1.3}$$

It follows that r^2 is an even integer. Therefore, r is an even integer (see Exercise 1.4.1). We can then write $r = 2j$ for some integer j. Hence $r^2 = 4j^2$. Substituting in (1.3), we get $s^2 = 2j^2$. Therefore, s^2 is even. We conclude as before that s is even. Thus, both r and s have a common factor, which is a contradiction. $\square$

The next theorem shows that irrational numbers are as ubiquitous as rational numbers.

Theorem 1.6.5 Let x and y be two real numbers such that $x < y$. Then there exists an irrational number t such that

$$x < t < y.$$

Proof: Since $x < y$, one has

$$x - \sqrt{2} < y - \sqrt{2}$$

By Theorem 1.6.3, there exists a rational number r such that

$$x - \sqrt{2} < r < y - \sqrt{2}$$

This implies

$$x < r + \sqrt{2} < y.$$

Since r is rational, the number $t = r + \sqrt{2}$ is irrational (see Exercise 1.6.4) and $x < t < y$. $\square$

Exercises

1.6.1 For each sets below determine if it is bounded above, bounded below, or both. If it is bounded above (below) find the supremum (infimum). Justify all your conclusions.

(a) $\left\{\dfrac{3n}{n+4} : n \in \mathbb{N}\right\}$

(b) $\left\{(-1)^n + \dfrac{1}{n} : n \in \mathbb{N}\right\}$

(c) $\left\{(-1)^n - \dfrac{(-1)^n}{n} : n \in \mathbb{N}\right\}$

1.6.2 ▶ Let r be a rational number such that $0 < r < 1$. Prove that there is $n \in \mathbb{N}$ such that

$$\frac{1}{n+1} < r \le \frac{1}{n}.$$

1.6.3 Let $x \in \mathbb{R}$. Prove that for every $n \in \mathbb{N}$, there is $r \in \mathbb{Q}$ such that $|x - r| < \dfrac{1}{n}$.

1.6.4 Prove that if x is a rational number and y is an irrational number, then $x+y$ is irrational. What can you say about xy?

1.6.5 Prove that in between two real numbers a and b with $a<b$, there are infinitely many rational numbers.

1.6.6 Prove that in between two real numbers a and b with $a<b$, there are infinitely many irrational numbers.

2. SEQUENCES

We introduce the notion of limit first through sequences. As mentioned in Chapter 1, a sequence is just a function with domain $\mathbb{N}$. More precisely, a sequence of elements of a set A is a function $f: \mathbb{N} \to A$. We will denote the image of n under the function with subscripted variables, for example, $a_n = f(n)$. We will also denote sequences by $\{a_n\}_{n=1}^{\infty}$, $\{a_n\}_n$, or even $\{a_n\}$. Each value a_n is called a term of the sequence, more precisely, the n-th term of the sequence.

■ **Example 2.0.1** Consider the sequence $a_n = \frac{1}{n}$ for $n \in \mathbb{N}$. This is a sequence of rational numbers. On occasion, when the pattern is clear, we may list the terms explicitly as in

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

■ **Example 2.0.2** Let $a_n = (-1)^n$ for $n \in \mathbb{N}$. This is a sequence of integers, namely,

$$-1, 1, -1, 1, -1, 1, \ldots$$

Note that the sequence takes on only two values. This should not be confused with the two-element set $\{1, -1\}$.

2.1 CONVERGENCE

Definition 2.1.1 Let $\{a_n\}$ be a sequence of real numbers. We say that the sequence $\{a_n\}$ *converges* to $a \in \mathbb{R}$ if, for any $\varepsilon > 0$, there exists a positive integer N such that for any $n \in \mathbb{N}$ with $n \geq N$, one has

$$|a_n - a| < \varepsilon \text{ (or equivalently, } a - \varepsilon < a_n < a + \varepsilon).$$

In this case, we call a the *limit of the sequence* (see Theorem 2.1.3 below) and write $\lim_{n\to\infty} a_n = a$. If the sequence $\{a_n\}$ does not converge, we call the sequence *divergent*.

Remark 2.1.1 It follows directly from the definition, using the Archimedean property, that a sequence $\{a_n\}$ converges to a if and only if for any $\varepsilon > 0$, there exists a real number N such that for any $n \in \mathbb{N}$ with $n > N$, one has

$$|a_n - a| < \varepsilon.$$

■ **Example 2.1.1** Let $a_n = \frac{1}{n}$ for $n \in \mathbb{N}$. We claim that $\lim_{n\to\infty} a_n = 0$. We verify it using the definition. Let $\varepsilon > 0$. Choose an integer $N > 1/\varepsilon$. (Note that such an integer N exists due to the Archimidean Property.) Then, if $n \geq N$, we get

$$|a_n - 0| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \frac{1}{1/\varepsilon} = \varepsilon.$$

■ **Example 2.1.2** We now generalize the previous example as follows. Let $\alpha > 0$ and consider the sequence given by

$$a_n = \frac{1}{n^\alpha} \text{ for } n \in \mathbb{N}.$$

Let $\varepsilon > 0$. Choose an integer N such that $N > (\frac{1}{\varepsilon})^{1/\alpha}$. For every $n \geq N$, one has $n > (\frac{1}{\varepsilon})^{1/\alpha}$ and, hence, $n^\alpha > \frac{1}{\varepsilon}$. This implies

$$\left|\frac{1}{n^\alpha} - 0\right| = \frac{1}{n^\alpha} < \frac{1}{1/\varepsilon} = \varepsilon.$$

We conclude that $\lim_{n\to\infty} a_n = 0$.

■ **Example 2.1.3** Consider the sequence $\{a_n\}$ where

$$a_n = \frac{3n^2+4}{2n^2+n+5}.$$

We will prove directly from the definition that this sequence converges to $a = \frac{3}{2}$.

Let $\varepsilon > 0$. We first search for a suitable N. To that end, we simplify and estimate the expression $|a_n - a|$. Notice that

$$\begin{aligned}\left|a_n - \frac{3}{2}\right| &= \left|\frac{3n^2+4}{2n^2+n+5} - \frac{3}{2}\right| = \left|\frac{2(3n^2+4) - 3(2n^2+n+5)}{2(2n^2+n+5)}\right| = \left|\frac{-7-3n}{2(2n^2+n+5)}\right| \\ &= \frac{3n+7}{2(2n^2+n+5)} < \frac{10n}{4n^2} = \frac{10}{4n}.\end{aligned}$$

To guarantee that the last expression is less than ε, it will suffice to choose $N > \dfrac{10}{4\varepsilon}$. Indeed, if $n \geq N$, we get

$$|a_n - a| \leq \frac{10}{4n} \leq \frac{10}{4N} < \frac{10}{4\frac{10}{4\varepsilon}} = \varepsilon.$$

■ **Example 2.1.4** Let $\{a_n\}$ be given by

$$a_n = \frac{4n^2-1}{3n^2-n}.$$

We claim $\lim_{n\to\infty]} a_n = \frac{4}{3}$.

Let $\varepsilon > 0$. We search for a suitable N. First notice that

$$\left|\frac{4n^2-1}{3n^2-n} - \frac{4}{3}\right| = \left|\frac{12n^2-3-12n^2+4n}{3(3n^2-n)}\right| = \left|\frac{4n-3}{3(3n^2-n)}\right|.$$

Since $n \geq 1$, we have $n^2 \geq n$ and $4n > 3$. Therefore we have

$$\left|\frac{4n^2-1}{3n^2-n}-\frac{4}{3}\right| = \frac{4n-3}{3(3n^2-n)} \leq \frac{4n-3}{3(3n^2-n^2)} < \frac{4n}{6n^2} = \frac{4}{6n}.$$

Thus, if $N > \frac{4}{6\varepsilon}$, we have, for $n \geq N$

$$\left|\frac{4n^2-1}{3n^2-n}-\frac{4}{3}\right| \leq \frac{4}{6n} \leq \frac{4}{6N} < \varepsilon.$$

■ **Example 2.1.5** Consider the sequence given by

$$a_n = \frac{n^2+5}{4n^2+n}.$$

We prove directly from the definition that $\{a_n\}$ converges to $\frac{1}{4}$.

Let $\varepsilon > 0$. Now,

$$\left|\frac{n^2+5}{4n^2+n}-\frac{1}{4}\right| = \left|\frac{4n^2+20-4n^2-n}{4(4n^2+n)}\right| = \frac{|20-n|}{4(4n^2+n)}.$$

If $n \geq 20$, then $|20-n| = n-20$. Therefore, for such n we have

$$\left|\frac{n^2+5}{4n^2+n}-\frac{1}{4}\right| = \frac{n-20}{4(4n^2+n)} \leq \frac{n}{16n^2} = \frac{1}{16n}.$$

Choose $N > \max\{\frac{1}{16\varepsilon}, 20\}$. Then, for $n \geq N$ we get

$$\left|\frac{n^2+5}{4n^2+n}-\frac{1}{4}\right| \leq \frac{1}{16n} \leq \frac{1}{16N} < \varepsilon.$$

The following result is quite useful in proving certain inequalities between numbers.

Lemma 2.1.2 Let $\ell \geq 0$. If $\ell < \varepsilon$ for all $\varepsilon > 0$, then $\ell = 0$.

Proof: This is easily proved by contraposition. If $\ell > 0$, then there is a positive number, for example $\varepsilon = \ell/2$, such that $\varepsilon < \ell$. □

Theorem 2.1.3 A convergent sequence $\{a_n\}$ has at most one limit.

Proof: Suppose $\{a_n\}$ converges to a and b. Then given $\varepsilon > 0$, there exist positive integers N_1 and N_2 such that

$$|a_n - a| < \varepsilon/2 \text{ for all } n \geq N_1$$

and

$$|a_n - b| < \varepsilon/2 \text{ for all } n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then

$$|a-b| \leq |a-a_N| + |a_N - b| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, by Lemma 2.1.2, $|a-b| = 0$ and, hence, $a = b$. □

The following lemma is a simple generalization of (2.1.2).

Lemma 2.1.4 Given real numbers a,b, then $a \leq b$ if and only if $a < b+\varepsilon$ for all $\varepsilon > 0$.

Proof: Suppose $a < b+\varepsilon$ for all $\varepsilon > 0$. And suppose, by way of contradiction, that $a > b$. then set $\varepsilon_0 = a-b$. Then $\varepsilon_0 > 0$. By assumption, we should have $a < b+\varepsilon_0 = b+a-b = a$, which is a contradiction. It follows that $a \leq b$.

The other direction follows immediately from the order axioms. □

The following comparison theorem shows that (non-strict) inequalities are preserved “in the limit”.

Theorem 2.1.5 — Comparison Theorem. Suppose $\{a_n\}$ and $\{b_n\}$ converge to a and b, respectively, and $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then $a \leq b$.

Proof: For any $\varepsilon > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$a - \frac{\varepsilon}{2} < a_n < a + \frac{\varepsilon}{2}, \quad \text{for } n \geq N_1,$$
$$b - \frac{\varepsilon}{2} < b_n < b + \frac{\varepsilon}{2}, \quad \text{for } n \geq N_2.$$

Choose $N > \max\{N_1, N_2\}$. Then

$$a - \frac{\varepsilon}{2} < a_N \leq b_N < b + \frac{\varepsilon}{2}.$$

Thus, $a < b+\varepsilon$ for any $\varepsilon > 0$. Using Lemma 2.1.4 we conclude $a \leq b$. □

Theorem 2.1.6 — The Squeeze Theorem. Suppose the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ satisfy

$$a_n \leq b_n \leq c_n \text{ for all } n \in \mathbb{N},$$

and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \ell$. Then $\lim_{n\to\infty} b_n = \ell$.

Proof: Fix any $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = \ell$, there exists $N_1 \in \mathbb{N}$ such that

$$\ell - \varepsilon < a_n < \ell + \varepsilon$$

for all $n \geq N_1$. Similarly, since $\lim_{n\to\infty} c_n = \ell$, there exists $N_2 \in \mathbb{N}$ such that

$$\ell - \varepsilon < c_n < \ell + \varepsilon$$

for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then, for $n \geq N$, we have

$$\ell - \varepsilon < a_n \leq b_n \leq c_n < \ell + \varepsilon,$$

which implies $|b_n - \ell| < \varepsilon$. Therefore, $\lim_{n\to\infty} b_n = \ell$. □

Definition 2.1.2 A sequence $\{a_n\}$ is *bounded above* if the set $\{a_n : n \in \mathbb{N}\}$ is bounded above. Similarly, the sequence $\{a_n\}$ is *bounded below* if the set $\{a_n : n \in \mathbb{N}\}$ is bounded below. We say that the sequence $\{a_n\}$ is *bounded* if the set $\{a_n : n \in \mathbb{N}\}$ is bounded, that is, if it is both bounded above and bounded below.

It follows from the observation after Definition 1.5.1 that the sequence $\{a_n\}$ is bounded if and only if there is $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem 2.1.7 A convergent sequence is bounded.

Proof: Suppose the sequence $\{a_n\}$ converges to a. Then, for $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a| < 1 \text{ for all } n \geq N.$$

Since $|a_n| - |a| \leq ||a_n| - |a|| \leq |a_n - a|$, this implies $|a_n| < 1 + |a|$ for all $n \geq N$. Set

$$M = \max\{|a_1|, \ldots, |a_{N-1}|, |a| + 1\}.$$

Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. Therefore, $\{a_n\}$ is bounded. □

Definition 2.1.3 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. The sequence $\{b_n\}_{n=1}^{\infty}$ is called a *subsequence* of $\{a_n\}_{n=1}^{\infty}$ if there exists a sequence of increasing positive integers

$$n_1 < n_2 < n_3 < \cdots,$$

such that $b_k = a_{n_k}$ for each $k \in \mathbb{N}$.

■ **Example 2.1.6** Consider the sequence $a_n = (-1)^n$ for $n \in \mathbb{N}$.

Then $\{a_{2k}\}$ is a subsequence of $\{a_n\}$ and $a_{2k} = 1$ for all k (here $n_k = 2k$ for all k). Similarly, $\{a_{2k+1}\}$ is also a subsequence of $\{a_n\}$ and $a_{2k+1} = -1$ for all k (here $n_k = 2k + 1$ for all k).

Lemma 2.1.8 Let $\{n_k\}_k$ be a sequence of positive integers with

$$n_1 < n_2 < n_3 < \cdots$$

Then $n_k \geq k$ for all $k \in \mathbb{N}$.

Proof: We use mathematical induction. When $k = 1$, it is clear that $n_1 \geq 1$ since n_1 is a positive integer. Assume $n_k \geq k$ for some k. Now $n_{k+1} > n_k$ and, since n_k and n_{k+1} are integers, this implies, $n_{k+1} \geq n_k + 1$. Therefore, $n_{k+1} \geq k + 1$ by the inductive hypothesis. The conclusion now follows by the principle of mathematical induction. □

Theorem 2.1.9 If a sequence $\{a_n\}$ converges to a, then any subsequence $\{a_{n_k}\}$ of $\{a_n\}$ also converges to a.

Proof: Suppose $\{a_n\}$ converges to a and let $\varepsilon > 0$ be given. Then there exists N such that

$$|a_n - a| < \varepsilon \text{ for all } n \geq N.$$

For any $k \geq N$, since $n_k \geq k$, we also have

$$|a_{n_k} - a| < \varepsilon.$$

Thus, $\{a_{n_k}\}$ converges to a as $k \to \infty$. □

■ **Example 2.1.7** Let $a_n = (-1)^n$ for $n \in \mathbb{N}$. Then the sequence $\{a_n\}$ is divergent. Indeed, suppose by contradiction that

$$\lim_{n\to\infty} a_n = \ell.$$

Then every subsequence of $\{a_n\}$ converges to a number $\ell \in \mathbb{R}$. From the previous theorem, it follows, in particular, that

$$\ell = \lim_{k\to\infty} a_{2k} = 1 \text{ and } \ell = \lim_{k\to\infty} a_{2k+1} = -1.$$

This contradiction shows that the sequence is divergent.

Since the sequence $\{a_n\}$ is bounded but not convergent, this example illustrates the fact that the converse of theorem 2.1.7 is not true.

Remark 2.1.10 Given a positive integer k_0, it will be convenient to also talk about the sequence $\{a_n\}_{n\geq k_0}$, that is, a function defined only for the integers greater than or equal to k_0. For simplicity of notation, we may also denote this sequence by $\{a_n\}$ whenever the integer k_0 is clear from the context. For instance, we talk of the sequence $\{a_n\}$ given by

$$a_n = \frac{n+1}{(n-1)(n-2)}.$$

although a_1 and a_2 are not defined. In all cases, the sequence must be defined from some integer onwards.

Exercises

2.1.1 Prove the following directly from the definition of limit.

(a) $\lim_{n\to\infty} \dfrac{2n^2+2}{3n^3+1} = 0.$

(b) $\lim_{n\to\infty} \dfrac{n^2+1}{5n^2+n+1} = \frac{1}{5}.$

(c) $\lim_{n\to\infty} \dfrac{2n^3+1}{4n^3-n} = \frac{1}{2}.$

(d) $\lim_{n\to\infty} \dfrac{3n^2+5}{6n^2+n} = \dfrac{1}{2}.$

(e) $\lim_{n\to\infty} \dfrac{4n^2-1}{n^2-n} = 4.$

2.1.2 Prove that if $\{a_n\}$ is a convergent sequence, then $\{|a_n|\}$ is a convergent sequence. Is the converse true?

2.1.3 Let $\{a_n\}$ be a sequence. Prove that if the sequence $\{|a_n|\}$ converges to 0, then $\{a_n\}$ also converges to 0.

2.1.4 Prove that $\lim_{n\to\infty} \dfrac{\sin n}{n} = 0.$

2.1.5 Let $\{x_n\}$ be a bounded sequence and let $\{y_n\}$ be a sequence that converges to 0. Prove that the sequence $\{x_n y_n\}$ converges to 0.

2.1.6 Prove that the following limits are 0. (*Hint:* use Theorem 2.1.6.)

(a) $\displaystyle\lim_{n\to\infty} \frac{n+\cos(n^2-3)}{2n^2+1}.$

(b) $\lim_{n\to\infty} \frac{3^n}{n!}$.

(c) $\lim_{n\to\infty} \frac{n!}{n^n}$.

(d) $\lim_{n\to\infty} \frac{n^2}{3^n}$. (*Hint:* see Exercise 1.3.4(c)).

2.1.7 Prove that if $\lim_{n\to\infty} a_n = \ell > 0$, then there exists $N \in \mathbb{N}$ such that $a_n > 0$ for all $n \geq N$.

2.1.8 ▶ Prove that if $\lim_{n\to\infty} a_n = \ell \neq 0$, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Is the conclusion still true if $\ell = 0$?

2.1.9 Let $\{a_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = 3$. Use Definition 2.1.1 to prove the following

(a) $\lim_{n\to\infty} 3a_n - 7 = 2$;

(b) $\lim_{n\to\infty} \frac{a_n + 1}{a_n} = \frac{4}{3}$; (*Hint:* prove first that there is N such that $a_n > 1$ for$n \geq N$.)

2.1.10 Let $a_n \geq 0$ for all $n \in \mathbb{N}$. Prove that if $\lim_{n\to\infty} a_n = \ell$, then $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{\ell}$.

2.1.11 Prove that the sequence $\{a_n\}$ with $a_n = \sin(n\pi/2)$ is divergent.

2.1.12 ▶ Consider a sequence $\{a_n\}$.

(a) Prove that $\lim_{n\to\infty} a_n = \ell$ if and only if $\lim_{k\to\infty} a_{2k} = \ell$ and $\lim_{k\to\infty} a_{2k+1} = \ell$.

(b) Prove that $\lim_{n\to\infty} a_n = \ell$ if and only if $\lim_{k\to\infty} a_{3k} = \ell$, $\lim_{k\to\infty} a_{3k+1} = \ell$, and $\lim_{k\to\infty} a_{3k+2} = \ell$.

2.1.13 Given a sequence $\{a_n\}$, define a new sequence $\{b_n\}$ by

$$b_n = \frac{a_1 + a_2 + \ldots + a_n}{n}.$$

(a) Prove that if $\lim_{n\to\infty} a_n = \ell$, then $\lim_{n\to\infty} b_n = \ell$.

(b) Find a counterexample to show that the converse does not hold in general.

2.2 LIMIT THEOREMS

We now prove several theorems that facilitate the computation of limits of some sequences in terms of those of other simpler sequences.

Theorem 2.2.1 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let k be a real number. Suppose $\{a_n\}$ converges to a and $\{b_n\}$ converges to b. Then the sequences $\{a_n + b_n\}$, $\{ka_n\}$, and $\{a_n b_n\}$ converge and

(a) $\lim_{n\to\infty}(a_n + b_n) = a + b$;

(b) $\lim_{n\to\infty}(ka_n) = ka$;

(c) $\lim_{n\to\infty}(a_n b_n) = ab$;

(d) If in addition $b \neq 0$ and $b_n \neq 0$ for $n \in \mathbb{N}$, then $\left\{\frac{a_n}{b_n}\right\}$ converges and $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

Proof: (a) Fix any $\varepsilon > 0$. Since $\{a_n\}$ converges to a, there exists $N_1 \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\varepsilon}{2} \text{ for all } n \geq N_1.$$

Similarly, there exists $N_2 \in \mathbb{N}$ such that

$$|b_n - b| < \frac{\varepsilon}{2} \text{ for all } n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. For any $n \geq N$, one has

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $\lim_{n\to\infty}(a_n + b_n) = a + b$. This proves (a).

(b) If $k = 0$, then $ka = 0$ and $ka_n = 0$ for all n. The conclusion follows immediately. Suppose next that $k \neq 0$. Given $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that $|a_n - a| < \frac{\varepsilon}{|k|}$ for $n \geq N$. Then for $n \geq N$, $|ka_n - ka| = |k||a_n - a| < \varepsilon$. It follows that $\lim_{n\to\infty}(ka_n) = ka$ as desired. This proves (b).

(c) Since $\{a_n\}$ is convergent, it follows from Theorem 2.1.7 that it is bounded. Thus, there exists $M > 0$ such that

$$|a_n| \leq M \text{ for all } n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$, we have the following estimate:

$$|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab| \leq |a_n||b_n - b| + |b||a_n - a|. \tag{2.1}$$

Let $\varepsilon > 0$. Since $\{a_n\}$ converges to a, we may choose $N_1 \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\varepsilon}{2(|b| + 1)} \text{ for all } n \geq N_1.$$

Similarly, since $\{b_n\}$ converges to b, we may choose $N_2 \in \mathbb{N}$ such that

$$|b_n - b| < \frac{\varepsilon}{2M} \text{ for all } n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then, for $n \geq N$, it follows from (2.1) that

$$|a_n b_n - ab| < M\frac{\varepsilon}{2M} + |b|\frac{\varepsilon}{2(|b| + 1)} < \varepsilon \text{ for all } n \geq N.$$

Therefore, $\lim_{n\to\infty} a_n b_n = ab$. This proves (c).

(d) Let us first show that

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}.$$

Since $\{b_n\}$ converges to b, there is $N_1 \in \mathbb{N}$ such that

$$|b_n - b| < \frac{|b|}{2} \text{ for } n \geq N_1.$$

It follows (using a triangle inequality) that, for such n, $-\frac{|b|}{2} < |b_n| - |b| < \frac{|b|}{2}$ and, hence, $\frac{|b|}{2} < |b_n|$. For each $n \geq N_1$, we have the following estimate

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b_n - b|}{|b_n||b|} \leq \frac{2|b_n - b|}{b^2}. \tag{2.2}$$

Now let $\varepsilon > 0$. Since $\lim_{n\to\infty} b_n = b$, there exists $N_2 \in \mathbb{N}$ such that

$$|b_n - b| < \frac{b^2\varepsilon}{2} \text{ for all } n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. By (2.2), one has

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| \leq \frac{2|b_n - b|}{b^2} < \varepsilon \text{ for all } n \geq N.$$

It follows that $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$.

Finally, we can apply part (c) and have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \frac{1}{b_n} = \frac{a}{b}.$$

The proof is now complete. □

■ **Example 2.2.1** Consider the sequence $\{a_n\}$ given by

$$a_n = \frac{3n^2 - 2n + 5}{1 - 4n + 7n^2}. \tag{2.3}$$

Dividing numerator and denominator by n^2, we can write

$$a_n = \frac{3 - 2/n + 5/n^2}{1/n^2 - 4/n + 7} \tag{2.4}$$

Therefore, by the limit theorems above,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3 - 2/n + 5/n^2}{1/n^2 - 4/n + 7} = \frac{\lim_{n\to\infty} 3 - \lim_{n\to\infty} 2/n + \lim_{n\to\infty} 5/n^2}{\lim_{n\to\infty} 1/n^2 - \lim_{n\to\infty} 4/n + \lim_{n\to\infty} 7} = \frac{3}{7}. \tag{2.5}$$

■ **Example 2.2.2** Let $a_n = \sqrt[n]{b}$, where $b > 0$. Consider the case where $b > 1$. In this case, $a_n > 1$ for every n. By the binomial theorem,

$$b = a_n^n = (a_n - 1 + 1)^n \geq 1 + n(a_n - 1).$$

This implies

$$0 < a_n - 1 \leq \frac{b-1}{n}.$$

For each $\varepsilon > 0$, choose $N > \frac{b-1}{\varepsilon}$. It follows that for $n \geq N$,

$$|a_n - 1| = a_n - 1 < \frac{b-1}{n} \leq \frac{b-1}{N} < \varepsilon.$$

Thus, $\lim_{n\to\infty} a_n = 1$.

In the case where $b = 1$, it is obvious that $a_n = 1$ for all n and, hence, $\lim_{n\to\infty} a_n = 1$.

If $0 < b < 1$, let $c = \frac{1}{b}$ and define

$$x_n = \sqrt[n]{c} = \frac{1}{a_n}.$$

Since $c > 1$, it has been shown that $\lim_{n\to\infty} x_n = 1$. This implies

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{x_n} = 1.$$

Exercises

2.2.1 Find the following limits:

(a) $\lim\limits_{n\to\infty} \dfrac{3n^2 - 6n + 7}{4n^2 - 3}$,

(b) $\lim\limits_{n\to\infty} \dfrac{1 + 3n - n^3}{3n^3 - 2n^2 + 1}$.

2.2.2 Find the following limits:

(a) $\lim\limits_{n\to\infty} \dfrac{\sqrt{3n}+1}{\sqrt{n}+\sqrt{3}}$,

(b) $\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{2n+1}{n}}$.

2.2.3 ▶ Find the following limits if they exist:

(a) $\lim_{n\to\infty}(\sqrt{n^2+n} - n)$.

(b) $\lim_{n\to\infty}(\sqrt[3]{n^3+3n^2} - n)$.

(c) $\lim_{n\to\infty}(\sqrt[3]{n^3+3n^2} - \sqrt{n^2+n})$.

(d) $\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n})$.

(e) $\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n})/n$.

2.2.4 Find the following limits.

(a) For $|r| < 1$ and $b \in \mathbb{R}$, $\lim_{n\to\infty}(b + br + br^2 + \cdots + br^n)$.

(b) $\lim_{n\to\infty}\left(\dfrac{2}{10} + \dfrac{2}{10^2} + \cdots + \dfrac{2}{10^n}\right)$.

2.2.5 Prove or disprove the following statements:

(a) If $\{a_n\}$ and $\{b_n\}$ are convergent sequences, then $\{a_n + b_n\}$ is a convergent sequence.
(b) If $\{a_n\}$ and $\{b_n\}$ are divergent sequences, then $\{a_n + b_n\}$ is divergent sequence.
(c) If $\{a_n\}$ and $\{b_n\}$ are convergent sequences, then $\{a_n b_n\}$ is a convergent sequence.
(d) If $\{a_n\}$ and $\{b_n\}$ are divergent sequences, then $\{a_n b_n\}$ is a divergent sequence.
(e) If $\{a_n\}$ and $\{a_n + b_n\}$ are convergent sequences, then $\{b_n\}$ is a convergent sequence.
(f) If $\{a_n\}$ and $\{a_n + b_n\}$ are divergent sequences, then $\{b_n\}$ is a divergent sequence.

2.3 MONOTONE SEQUENCES

Definition 2.3.1 A sequence $\{a_n\}$ is called *increasing* if

$$a_n \le a_{n+1} \text{ for all } n \in \mathbb{N}.$$

It is called *decreasing* if

$$a_n \ge a_{n+1} \text{ for all } n \in \mathbb{N}.$$

If $\{a_n\}$ is increasing or decreasing, then it is called a *monotone* sequence.

The sequence is called *strictly* increasing (resp. *strictly* decreasing) if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$ (resp. $a_n > a_{n+1}$ for all $n \in \mathbb{N}$).

It is easy to show by induction that if $\{a_n\}$ is an increasing sequence, then $a_n \le a_m$ whenever $n \le m$.

Theorem 2.3.1 — Monotone Convergence Theorem. Let $\{a_n\}$ be a sequence of real numbers. The following hold:

(a) If $\{a_n\}$ is increasing and bounded above, then it is convergent.
(b) If $\{a_n\}$ is decreasing and bounded below, then it is convergent.

Proof: (a) Let $\{a_n\}$ be an increasing sequence that is bounded above. Define

$$A = \{a_n : n \in \mathbb{N}\}.$$

Then A is a subset of $\mathbb{R}$ that is nonempty and bounded above and, hence, $\sup A$ exists. Let $\ell = \sup A$ and let $\varepsilon > 0$. By Proposition 1.5.1, there exists $N \in \mathbb{N}$ such that

$$\ell - \varepsilon < a_N \le \ell.$$

Since $\{a_n\}$ is increasing,

$$\ell - \varepsilon < a_N \le a_n \text{ for all } n \ge N.$$

On the other hand, since ℓ is an upper bound for A, we have $a_n \le \ell$ for all n. Thus,

$$\ell - \varepsilon < a_n < \ell + \varepsilon \text{ for all } n \ge N.$$

Therefore, $\lim_{n\to\infty} a_n = \ell$.

(b) Let $\{a_n\}$ be a decreasing sequence that is bounded below. Define

$$b_n = -a_n.$$

Then $\{b_n\}$ is increasing and bounded above (if M is a lower bound for $\{a_n\}$, then $-M$ is an upper bound for $\{b_n\}$). Let

$$\ell = \lim_{n\to\infty} b_n = \lim_{n\to\infty}(-a_n).$$

Then $\{a_n\}$ converges to $-\ell$ by Theorem 2.2.1. □

Remark 2.3.2 It follows from the proof of Theorem 2.3.1 that if $\{a_n\}$ is increasing and bounded above, then

$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$

Similarly, if $\{a_n\}$ is decreasing and bounded below, then

$$\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}.$$

■ **Example 2.3.1** Given $r \in \mathbb{R}$ with $|r| < 1$, define $a_n = r^n$ for $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} a_n = 0.$$

This is clear if $r = 0$. Let us first consider the case where $0 < r < 1$. Then $0 \le a_{n+1} = ra_n \le a_n$ for all n. Therefore, $\{a_n\}$ is decreasing and bounded below. By Theorem 2.3.1, the sequence converges. Let

$$\ell = \lim_{n\to\infty} a_n.$$

Since $a_{n+1} = ra_n$ for all n, taking limits on both sides gives $\ell = r\ell$. Thus, $(1-r)\ell = 0$ and, hence, $\ell = 0$. In the general case, we only need to consider the sequence defined by $b_n = |a_n|$ for $n \in \mathbb{N}$; see Exercise 2.1.3.

■ **Example 2.3.2** Consider the sequence $\{a_n\}$ defined as follows:

$$a_1 = 2 \tag{2.6}$$

$$a_{n+1} = \frac{a_n + 5}{3} \text{ for } n \ge 1. \tag{2.7}$$

First we will show that the sequence is increasing. We prove by induction that for all $n \in \mathbb{N}$, $a_n < a_{n+1}$. Since $a_2 = \frac{a_1+5}{3} = \frac{7}{3} > 2 = a_1$, the statement is true for $n = 1$. Next, suppose $a_k < a_{k+1}$ for some $k \in \mathbb{N}$. Then $a_k + 5 < a_{k+1} + 5$ and $(a_k + 5)/3 < (a_{k+1} + 5)/3$. Therefore,

$$a_{k+1} = \frac{a_k + 5}{3} < \frac{a_{k+1} + 5}{3} = a_{k+2}.$$

It follows by induction that the sequence is increasing.

Next we prove that the sequence is bounded by 3. Again, we proceed by induction. The statement is clearly true for $n = 1$. Suppose that $a_k \le 3$ for some $k \in \mathbb{N}$. Then

$$a_{k+1} = \frac{a_k + 5}{3} \le \frac{3+5}{3} = \frac{8}{3} \le 3.$$

It follows that $a_n \le 3$ for all $n \in \mathbb{N}$.

From the Monotone Convergence Theorem, we deduce that there is $\ell \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = \ell$. Since the subsequence $\{a_{k+1}\}_{k=1}^{\infty}$ also converges to ℓ, taking limits on both sides of the equation in (2.7), we obtain

$$\ell = \frac{\ell+5}{3}.$$

Therefore, $3\ell = \ell+5$ and, hence, $\ell = 5/2$.

■ **Example 2.3.3 —The number e.** Consider the sequence $\{a_n\}$ given by

$$a_n = \left(1+\frac{1}{n}\right)^n, \; n \in \mathbb{N}.$$

By the binomial theorem,

$$\begin{aligned} a_n &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k \\ &= 1+1+\frac{n(n-1)}{2!}\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}\frac{1}{n^3}+\cdots+\frac{n(n-1)\cdots(n-(n-1))}{n!}\frac{1}{n^n} \\ &= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right). \end{aligned}$$

The corresponding expression for a_{n+1} has one more term and each factor $(1-\frac{k}{n})$ is replaced by the larger factor $(1-\frac{k}{n+1})$. It is then clear that $a_n < a_{n+1}$ for all $n \in \mathbb{N}$. Thus, the sequence is increasing. Moreover,

$$\begin{aligned} a_n &\le 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \\ &< 2+\frac{1}{1.2}+\frac{1}{2.3}+\cdots+\frac{1}{(n-1)\cdot n} \\ &= 2+\sum_{k=1}^{n-1}\left(\frac{1}{k}-\frac{1}{k+1}\right) = 3-\frac{1}{n} < 3. \end{aligned}$$

Hence the sequence is bounded above.

By the Monotone Convergence Theorem, $\lim_{n\to\infty} a_n$ exists and is denoted by e. In fact, e is an irrational number and $e \approx 2.71828$.

The following fundamental result is an application of the Monotone Convergence Theorem.

Theorem 2.3.3 — Nested Intervals Theorem. Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of nonempty closed bounded intervals satisfying $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$. Then the following hold:

(a) $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

(b) If, in addition, the lengths of the intervals I_n converge to zero, then $\bigcap_{n=1}^{\infty} I_n$ consists of a single point.

Proof: Let $\{I_n\}$ be as in the statement with $I_n = [a_n, b_n]$. In particular, $a_n \le b_n$ for all $n \in \mathbb{N}$. Given that $I_{n+1} \subset I_n$, we have $a_n \le a_{n+1}$ and $b_{n+1} \le b_n$ for all $n \in \mathbb{N}$. This shows that $\{a_n\}$ is an increasing sequence bounded above by b_1 and $\{b_n\}$ is a decreasing sequence bounded below by a_1. By the

Monotone Convergence Theorem (Theorem 2.3.1), there exist $a, b \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Since $a_n \leq b_n$ for all n, by Theorem 2.1.5, we get $a \leq b$. Now, we also have $a_n \leq a$ and $b \leq b_n$ for all $n \in \mathbb{N}$ (since $\{a_n\}$ is increasing and $\{b_n\}$ is decreasing). This shows that if $a \leq x \leq b$, then $x \in I_n$ for all $n \in \mathbb{N}$. Thus, $[a,b] \subset \bigcap_{n=1}^{\infty} I_n$. It follows that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. This proves part (a).

Now note also that $\bigcap_{n=1}^{\infty} I_n \subset [a,b]$. Indeed, if $x \in \bigcap_{n=1}^{\infty} I_n$, then $x \in I_n$ for all n. Therefore, $a_n \leq x \leq b_n$ for all n. Using Theorem 2.1.5, we conclude $a \leq x \leq b$. Thus, $x \in [a,b]$. This proves the desired inclusion and, hence, $\bigcap_{n=1}^{\infty} I_n = [a,b]$.

We now prove part (b). Suppose the lengths of the intervals I_n converge to zero. This means $b_n - a_n \to 0$ as $n \to \infty$. Then $b = \lim_{n\to\infty} b_n = \lim_{n\to\infty}[(b_n - a_n) + a_n] = a$. It follows that $\bigcap_{n=1}^{\infty} I_n = \{a\}$ as desired. $\square$

When a monotone sequence is not bounded, it does not converge. However, the behavior follows a clear pattern. To make this precise we provide the following definition.

Definition 2.3.2 A sequence $\{a_n\}$ is said to *diverge to* ∞ if for every $M \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that

$$a_n > M \text{ for all } n \geq N.$$

In this case, we write $\lim_{n\to\infty} a_n = \infty$. Similarly, we say that $\{a_n\}$ *diverges to* $-\infty$ and write $\lim_{n\to\infty} a_n = -\infty$ if for every $M \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that

$$a_n < M \text{ for all } n \geq N.$$

Remark 2.3.4 We should not confuse a sequence that diverges to ∞ (that is, one that satisfies the previous definition), with a divergent sequence (that is, one that does not converge).

■ **Example 2.3.4** Consider the sequence $a_n = \frac{n^2+1}{5n}$. We will show, using Definition 2.3.2, that $\lim_{n\to\infty} a_n = \infty$.

Let $M \in \mathbb{R}$. Note that

$$\frac{n^2+1}{5n} = \frac{n}{5} + \frac{1}{5n} \geq \frac{n}{5}.$$

Choose $N > 5M$. Then, if $n \geq N$, we have

$$a_n \geq \frac{n}{5} \geq \frac{N}{5} > M.$$

Th following result completes the description of the behavior of monotone sequences.

Theorem 2.3.5 If a sequence $\{a_n\}$ is increasing and not bounded above, then

$$\lim_{n\to\infty} a_n = \infty.$$

Similarly, if $\{a_n\}$ is decreasing and not bounded below, then

$$\lim_{n\to\infty} a_n = -\infty.$$

Proof: Fix any real number M. Since $\{a_n\}$ is not bounded above, there exists $N \in \mathbb{N}$ such that $a_N \geq M$. Then

$$a_n \geq a_N \geq M \text{ for all } n \geq N$$

because $\{a_n\}$ is increasing. Therefore, $\lim_{n\to\infty} a_n = \infty$. The proof for the second case is similar. $\square$

Theorem 2.3.6 Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers and let k be a constant. Suppose

$$\lim_{n\to\infty} a_n = \infty, \ \lim_{n\to\infty} b_n = \infty, \text{ and } \lim_{n\to\infty} c_n = -\infty$$

Then

(a) $\lim_{n\to\infty}(a_n + b_n) = \infty$;
(b) $\lim_{n\to\infty}(a_n b_n) = \infty$;
(c) $\lim_{n\to\infty}(a_n c_n) = -\infty$;
(d) $\lim_{n\to\infty} ka_n = \infty$ if $k > 0$, and $\lim_{n\to\infty} ka_n = -\infty$ if $k < 0$;
(e) $\lim_{n\to\infty} \dfrac{1}{a_n} = 0$. (Here we assume $a_n \neq 0$ for all n.)

Proof: We provide proofs for (a) and (e) and leave the others as exercises.

(a) Fix any $M \in \mathbb{R}$. Since $\lim_{n\to\infty} a_n = \infty$, there exists $N_1 \in \mathbb{N}$ such that

$$a_n \geq \frac{M}{2} \text{ for all } n \geq N_1.$$

Similarly, there exists $N_2 \in \mathbb{N}$ such that

$$b_n \geq \frac{M}{2} \text{ for all } n \geq N_1.$$

Let $N = \max\{N_1, N_2\}$. Then it is clear that

$$a_n + b_n \geq M \text{ for all } n \geq N.$$

This implies (a).

(e) For any $\varepsilon > 0$, let $M = \dfrac{1}{\varepsilon}$. Since $\lim_{n\to\infty} a_n = \infty$, there exists $N \in \mathbb{N}$ such that

$$a_n > \frac{1}{\varepsilon} \text{ for all } n \geq N.$$

This implies that for $n \geq N$,

$$\left|\frac{1}{a_n} - 0\right| = \frac{1}{a_n} < \varepsilon.$$

Thus, (e) holds. $\square$

The proof of the comparison theorem below follows directly from Definition 2.3.2 (see also Theorem 2.1.5).

Theorem 2.3.7 Suppose $a_n \leq b_n$ for all $n \in \mathbb{N}$.

(a) If $\lim_{n\to\infty} a_n = \infty$, then $\lim_{n\to\infty} b_n = \infty$.
(b) If $\lim_{n\to\infty} b_n = -\infty$, then $\lim_{n\to\infty} a_n = -\infty$.

Exercises

2.3.1 ▶ Let $a_1 = \sqrt{2}$. Define

$$a_{n+1} = \sqrt{a_n + 2} \text{ for } n \geq 1.$$

(a) Prove that $a_n < 2$ for all $n \in \mathbb{N}$.
(b) Prove that $\{a_n\}$ is an increasing sequence.
(c) Prove that $\lim_{n\to\infty} a_n = 2$.

2.3.2 ▷ Prove that each of the following sequences is convergent and find its limit.

(a) $a_1 = 1$ and $a_{n+1} = \dfrac{a_n + 3}{2}$ for $n \geq 1$.
(b) $a_1 = \sqrt{6}$ and $a_{n+1} = \sqrt{a_n + 6}$ for $n \geq 1$.
(c) $a_{n+1} = \dfrac{1}{3}\left(2a_n + \dfrac{1}{a_n^2}\right), n \geq 1, a_1 > 0$.
(d) $a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{b}{a_n}\right), b > 0$.

2.3.3 ▷ Prove that each of the following sequences is convergent and find its limit.

(a) $\sqrt{2}; \sqrt{2\sqrt{2}}; \sqrt{2\sqrt{2\sqrt{2}}}; \cdots$
(b) $1/2; \dfrac{1}{2+1/2}; \dfrac{1}{2+\dfrac{1}{2+1/2}}; \cdots$

2.3.4 Prove that the following sequence is convergent:

$$a_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}, \; n \in \mathbb{N}.$$

2.3.5 ▷ Let a and b be two positive real numbers with $a < b$. Define $a_1 = a$, $b_1 = b$, and

$$a_{n+1} = \sqrt{a_n b_n} \text{ and } b_{n+1} = \frac{a_n + b_n}{2} \text{ for } n \geq 1.$$

Show that $\{a_n\}$ and $\{b_n\}$ are convergent to the same limit.

2.3.6 Prove the following using Definition 2.3.2.

(a) $\displaystyle\lim_{n\to\infty} \frac{2n^2 + n + 1}{n - 2} = \infty$.
(b) $\displaystyle\lim_{n\to\infty} \frac{1 - 3n^2}{n + 2} = -\infty$.

2.3.7 Prove parts (b), (c), and (d) of Theorem 2.3.6.

2.3.8 Prove Theorem 2.3.7.

2.4 THE BOLZANO-WEIERSTRASS THEOREM

The Bolzano-Weierstrass Theorem is at the foundation of many results in analysis. It is, in fact, equivalent to the completeness axiom of the real numbers.

Theorem 2.4.1 — Bolzano-Weierstrass. Every bounded sequence $\{a_n\}$ of real numbers has a convergent subsequence.

Proof: Suppose $\{a_n\}$ is a bounded sequence. Define $A = \{a_n : n \in \mathbb{N}\}$ (the set of values of the sequence $\{a_n\}$). If A is finite, then at least one of the elements of A, say x, must be equal to a_n for infinitely many choices of n. More precisely, $B_x = \{n \in \mathbb{N} : a_n = x\}$ is infinite. We can then define a convergent subsequence as follows. Pick n_1 such that $a_{n_1} = x$. Now, since B_x is infinite, we can choose $n_2 > n_1$ such that $a_{n_2} = x$. Continuing in this way, we can define a subsequence $\{a_{n_k}\}$ which is constant, equal to x and, thus, converges to x.

Suppose now that A is infinite. First observe there exist $c, d \in \mathbb{R}$ such that $c \leq a_n \leq d$ for all $n \in \mathbb{N}$, that is, $A \subset [c, d]$.

We define a sequence of nonempty nested closed bounded intervals as follows. Set $I_1 = [c, d]$. Next consider the two subintervals $[c, \frac{c+d}{2}]$ and $[\frac{c+d}{2}, d]$. Since A is infinite, at least one of $A \cap [c, \frac{c+d}{2}]$ or $A \cap [\frac{c+d}{2}, d]$ is infinite. Let $I_2 = [c, \frac{c+d}{2}]$ if $A \cap [c, \frac{c+d}{2}]$ is infinite and $I_2 = [\frac{c+d}{2}, d]$ otherwise. Continuing in this way, we construct a nested sequence of nonempty closed bounded intervals $\{I_n\}$ such that $I_n \cap A$ is infinite and the length of I_n tends to 0 as $n \to \infty$.

We now construct the desired subsequence of $\{a_n\}$ as follows. Let $n_1 = 1$. Choose $n_2 > n_1$ such that $a_{n_2} \in I_2$. This is possible since $I_2 \cap A$ is infinite. Next choose $n_3 > n_2$ such that $a_{n_3} \in I_3$. In this way, we obtain a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \in I_k$ for all $k \in \mathbb{N}$.

Set $I_n = [c_n, d_n]$. Then $\lim_{n\to\infty}(d_n - c_n) = 0$. We also know from the proof of the Monotone Convergence Theorem (Theorem 2.3.1), that $\{c_n\}$ converges. Say $\ell = \lim_{n\to\infty} c_n$. Thus, $\lim_{n\to\infty} d_n = \lim_{n\to\infty}[(d_n - c_n) + c_n] = \ell$ as well. Since $c_k \leq a_{n_k} \leq d_k$ for all $k \in \mathbb{N}$, it follows from Theorem 2.1.5 that $\lim_{k\to\infty} a_{n_k} = \ell$. This completes the proof. □

Definition 2.4.1 (Cauchy sequence). A sequence $\{a_n\}$ of real numbers is called a *Cauchy sequence* if for any $\varepsilon > 0$, there exists a positive integer N such that for any $m, n \geq N$, one has

$$|a_m - a_n| < \varepsilon.$$

Theorem 2.4.2 A convergent sequence is a Cauchy sequence.

Proof: Let $\{a_n\}$ be a convergent sequence and let

$$\lim_{n\to\infty} a_n = a.$$

Then for any $\varepsilon > 0$, there exists a positive integer N such that

$$|a_n - a| < \varepsilon/2 \text{ for all } n \geq N.$$

For any $m, n \geq N$, one has

$$|a_m - a_n| \leq |a_m - a| + |a_n - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, $\{a_n\}$ is a Cauchy sequence. □

Theorem 2.4.3 A Cauchy sequence is bounded.

Proof: Let $\{a_n\}$ be a Cauchy sequence. Then for $\varepsilon = 1$, there exists a positive integer N such that

$$|a_m - a_n| < 1 \text{ for all } m, n \geq N.$$

In particular,

$$|a_n - a_N| < 1 \text{ for all } n \geq N.$$

Let $M = \max\{|a_1|, \dots, |a_{N-1}|, 1 + |a_N|\}$. Then, for $n = 1, \dots, N-1$, we clearly have $|a_n| \leq M$. Moreover, for $n \geq N$,

$$|a_n| = |a_n - a_N + a_N| \leq |a_n - a_N| + |a_N| \leq 1 + |a_N| \leq M.$$

Therefore, $|a_n| \leq M$ for all $n \in \mathbb{N}$ and, thus, $\{a_n\}$ is bounded. $\square$

Lemma 2.4.4 A Cauchy sequence that has a convergent subsequence is convergent.

Proof: Let $\{a_n\}$ be a Cauchy sequence that has a convergent subsequence. For any $\varepsilon > 0$, there exists a positive integer N such that

$$|a_m - a_n| \leq \varepsilon/2 \text{ for all } m, n \geq N.$$

Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$ that converges to some point a. For the above ε, there exists a positive number K such that

$$|a_{n_k} - a| < \varepsilon/2 \text{ for all } k \geq K.$$

Thus, we can find a positive integer $n_\ell > N$ such that

$$|a_{n_\ell} - a| < \varepsilon/2.$$

Then for any $n \geq N$, we have

$$|a_n - a| \leq |a_n - a_{n_\ell}| + |a_{n_\ell} - a| < \varepsilon.$$

Therefore, $\{a_n\}$ converges to a. $\square$

Theorem 2.4.5 Any Cauchy sequence of real numbers is convergent.

Proof: Let $\{a_n\}$ be a Cauchy sequence. Then it is bounded by Theorem 2.4.3. By the Bolzano-Weierstrass theorem, $\{a_n\}$ has a convergent subsequence. Therefore, it is convergent by Lemma 2.4.4. $\square$

Remark 2.4.6 It follows from Definition 2.4.1 that $\{a_n\}$ is a Cauchy sequence if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_{n+p} - a_n| < \varepsilon \text{ for all } n \geq N \text{ and for all } p \in \mathbb{N}.$$

Definition 2.4.2 A sequence $\{a_n\}$ is called contractive if there exists $k \in [0, 1)$ such that

$$|a_{n+2} - a_{n+1}| \leq k|a_{n+1} - a_n| \text{ for all } n \in \mathbb{N}.$$

Theorem 2.4.7 Every contractive sequence is convergent.

Proof: By induction, one has

$$|a_{n+1}-a_n| \leq k^{n-1}|a_2-a_1| \text{ for all } n \in \mathbb{N}.$$

Thus,

$$\begin{aligned} |a_{n+p}-a_n| &\leq |a_{n+1}-a_n|+|a_{n+2}-a_{n+1}|+\cdots+|a_{n+p}-a_{n+p-1}| \\ &\leq (k^{n-1}+k^n+\cdots+k^{n+p-2})|a_2-a_1| \\ &\leq k^{n-1}(1+k+k^2+\cdots+k^{p-1})|a_2-a_1| \\ &\leq \frac{k^{n-1}}{1-k}|a_2-a_1|. \end{aligned}$$

for all $n,p \in \mathbb{N}$. Since $k^{n-1} \to 0$ as $n \to \infty$ (independently of p), this implies $\{a_n\}$ is a Cauchy sequence and, hence, it is convergent. □

■ **Example 2.4.1** The condition $k<1$ in the previous theorem is crucial. Consider the following example. Let $a_n = \ln n$ for all $n \in \mathbb{N}$. Since $1 < \frac{n+2}{n+1} < \frac{n+1}{n}$ for all $n \in \mathbb{N}$ and the natural logarithm is an increasing function, we have

$$|a_{n+2}-a_{n+1}| = |\ln(n+2)-\ln(n+1)| = \left|\ln\left(\frac{n+2}{n+1}\right)\right| = \ln\left(\frac{n+2}{n+1}\right)$$

$$< \ln\left(\frac{n+1}{n}\right) = |\ln(n+1)-\ln n| = |a_{n+1}-a_n|.$$

Therefore, the inequality in Definition 2.4.2 is satisfied with $k=1$, yet the sequence $\{\ln n\}$ does not converge.

Exercises

2.4.1 ▶ Determine which of the following are Cauchy sequences.

(a) $a_n = (-1)^n$.
(b) $a_n = (-1)^n/n$.
(c) $a_n = n/(n+1)$.
(d) $a_n = (\cos n)/n$.

2.4.2 Prove that the sequence

$$a_n = \frac{n\cos(3n^2+2n+1)}{n+1}$$

has a convergent subsequence.

2.4.3 Let $f\colon [0,\infty) \to \mathbb{R}$ be such that $f(x) > 0$ for all x. Define

$$a_n = \frac{f(n)}{f(n)+1}.$$

Prove that the sequence a_n has a convergent subsequence.

2.4.4 Define

$$a_n = \frac{1+2^n}{2^n} \text{ for } n \in \mathbb{N}.$$

Prove that the sequence a_n is contractive.

2.4.5 Let $r \in \mathbb{R}$ be such that $|r| < 1$. Define $a_n = r^n$ for $n \in \mathbb{N}$. Prove that the sequence $\{a_n\}$ is contractive.

2.4.6 Prove that the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is not contractive.

2.5 LIMIT SUPERIOR AND LIMIT INFERIOR

We begin this section with a proposition which follows from Theorem 2.3.1. All sequences in this section are assumed to be of real numbers.

Proposition 2.5.1 Let $\{a_n\}$ be a bounded sequence. Define

$$s_n = \sup\{a_k : k \geq n\} \tag{2.8}$$

and

$$t_n = \inf\{a_k : k \geq n\}. \tag{2.9}$$

Then $\{s_n\}$ and $\{t_n\}$ are convergent.

Proof: If $n \leq m$, then $\{a_k : k \geq m\} \subset \{a_k : k \geq n\}$. Therefore, it follows from Theorem 1.5.3 that $s_n \geq s_m$ and, so, the sequence $\{s_n\}$ is decreasing. Since $\{a_n\}$ is bounded, then so is $\{s_n\}$. In particular, $\{s_n\}$ is bounded below. Similarly, $\{t_n\}$ is increasing and bounded above. Therefore, both sequences are convergent by Theorem 2.3.1. $\square$

Definition 2.5.1 Let $\{a_n\}$ be a sequence. Then the *limit superior* of $\{a_n\}$, denoted by $\limsup_{n\to\infty} a_n$, is defined by

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\}.$$

Note that $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} s_n$, where s_n is defined in (2.8).

Similarly, the *limit inferior* of $\{a_n\}$, denoted by $\liminf_{n\to\infty} a_n$, is defined by

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\}.$$

Note that $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} t_n$, where t_n is defined in (2.9).

Theorem 2.5.2 If $\{a_n\}$ is not bounded above, then

$$\lim_{n\to\infty} s_n = \infty,$$

where $\{s_n\}$ is defined in (2.8).

Similarly, if $\{a_n\}$ is not bounded below, then

$$\lim_{n\to\infty} t_n = -\infty,$$

where $\{t_n\}$ is defined in (2.9).

Proof: Suppose $\{a_n\}$ is not bounded above. Then for any $k \in \mathbb{N}$, the set $\{a_i : i \geq k\}$ is also not bounded above. Thus, $s_k = \sup\{a_i : i \geq k\} = \infty$ for all k. Therefore, $\lim_{k\to\infty} s_k = \infty$. The proof for the second case is similar. $\square$

Remark 2.5.3 By Theorem 2.5.2, we see that if $\{a_n\}$ is not bounded above, then

$$\limsup_{n\to\infty} a_n = \infty.$$

Similarly, if $\{a_n\}$ is not bounded below, then

$$\liminf_{n\to\infty} a_n = -\infty.$$

Theorem 2.5.4 Let $\{a_n\}$ be a sequence and $\ell \in \mathbb{R}$. The following are equivalent:

(a) $\limsup_{n\to\infty} a_n = \ell$.

(b) For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$a_n < \ell + \varepsilon \text{ for all } n \geq N,$$

and there exists a subsequence of $\{a_{n_k}\}$ of $\{a_n\}$ such that

$$\lim_{k\to\infty} a_{n_k} = \ell.$$

Proof: Suppose $\limsup_{n\to\infty} a_n = \ell$. Then $\lim_{n\to\infty} s_n = \ell$, where s_n is defined as in (2.8). For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\ell - \varepsilon < s_n < \ell + \varepsilon \text{ for all } n \geq N.$$

This implies $s_N = \sup\{a_n : n \geq N\} < \ell + \varepsilon$. Thus,

$$a_n < \ell + \varepsilon \text{ for all } n \geq N.$$

Moreover, for $\varepsilon = 1$, there exists $N_1 \in \mathbb{N}$ such that

$$\ell - 1 < s_{N_1} = \sup\{a_n : n \geq N_1\} < \ell + 1.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$\ell - 1 < a_{n_1} < \ell + 1.$$

For $\varepsilon = \frac{1}{2}$, there exists $N_2 \in \mathbb{N}$ and $N_2 > n_1$ such that

$$\ell - \frac{1}{2} < s_{N_2} = \sup\{a_n : n \geq N_2\} < \ell + \frac{1}{2}.$$

Thus, there exists $n_2 > n_1$ such that

$$\ell - \frac{1}{2} < a_{n_2} < \ell + \frac{1}{2}.$$

In this way, we can construct a strictly increasing sequence $\{n_k\}$ of positive integers such that

$$\ell - \frac{1}{k} < a_{n_k} < \ell + \frac{1}{k}.$$

Therefore, $\lim_{k\to\infty} a_{n_k} = \ell$.

We now prove the converse. Given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$a_n < \ell + \varepsilon \text{ and } \ell - \varepsilon < a_{n_k} < \ell + \varepsilon$$

for all $n \geq N$ and $k \geq N$. Let any $m \geq N$, we have

$$s_m = \sup\{a_k : k \geq m\} \leq \ell + \varepsilon.$$

By Lemma 2.1.8, $n_m \geq m$, so we also have

$$s_m = \sup\{a_k : k \geq m\} \geq a_{n_m} > \ell - \varepsilon.$$

Therefore, $\lim_{m\to\infty} s_m = \limsup_{m\to\infty} a_n = \ell$. $\square$

The following result is proved in a similar way.

Theorem 2.5.5 Let $\{a_n\}$ be a sequence and $\ell \in \mathbb{R}$. The following are equivalent:

(a) $\liminf_{n\to\infty} a_n = \ell$.

(b) For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$a_n > \ell - \varepsilon \text{ for all } n \geq N,$$

and there exists a subsequence of $\{a_{n_k}\}$ of $\{a_n\}$ such that

$$\lim_{k\to\infty} a_{n_k} = \ell.$$

The following corollary follows directly from Theorems 2.5.4 and 2.5.5.

Corollary 2.5.6 Let $\{a_n\}$ be a sequence. Then

$$\lim_{n\to\infty} a_n = \ell \text{ if and only if } \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \ell.$$

Corollary 2.5.7 Let $\{a_n\}$ be a sequence.

(a) Suppose $\limsup_{n\to\infty} a_n = \ell$ and $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ with

$$\lim_{k\to\infty} a_{n_k} = \ell'.$$

Then $\ell' \le \ell$.

(b) Suppose $\liminf_{n\to\infty} a_n = \ell$ and $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ with

$$\lim_{k\to\infty} a_{n_k} = \ell'.$$

Then $\ell' \ge \ell$.

Proof: We prove only (a) because the proof of (b) is similar. By Theorem 2.5.4 and the definition of limits, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$a_n < \ell + \varepsilon \text{ and } \ell' - \varepsilon < a_{n_k} < \ell' + \varepsilon$$

for all $n \ge N$ and $k \ge N$. Since $n_N \ge N$, this implies

$$\ell' - \varepsilon < a_{n_N} < \ell + \varepsilon.$$

Thus, $\ell' < \ell + 2\varepsilon$ and, hence, $\ell' \le \ell$ because ε is arbitrary. $\square$

Remark 2.5.8 Let $\{a_n\}$ be a bounded sequence. Define

$$A = \{x \in \mathbb{R} : \text{there exists a subsequence } \{a_{n_k}\} \text{ with } \lim a_{n_k} = x\}.$$

Each element of the set A called a *subsequential limit* of the sequence $\{a_n\}$. It follows from Theorem 2.5.4, Theorem 2.5.5, and Corollary 2.5.7 that $A \ne \emptyset$ and

$$\limsup_{n\to\infty} a_n = \max A \text{ and } \liminf_{n\to\infty} a_n = \min A.$$

Theorem 2.5.9 Suppose $\{a_n\}$ is a sequence such that $a_n > 0$ for every $n \in \mathbb{N}$ and

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell < 1.$$

Then $\lim_{n\to\infty} a_n = 0$.

Proof: Choose $\varepsilon > 0$ such that $\ell + \varepsilon < 1$. Then there exists $N \in \mathbb{N}$ such that

$$\frac{a_{n+1}}{a_n} < \ell + \varepsilon \text{ for all } n \ge N.$$

Let $q = \ell + \varepsilon$. Then $0 < q < 1$. By induction,

$$0 < a_n \le q^{n-N} a_N \text{ for all } n \ge N.$$

Since $\lim_{n\to\infty} q^{n-N} a_N = 0$, one has $\lim_{n\to\infty} a_n = 0$. $\square$

By a similar method, we obtain the theorem below.

Theorem 2.5.10 Suppose $\{a_n\}$ is a sequence such that $a_n > 0$ for every $n \in \mathbb{N}$ and

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell > 1.$$

Then $\lim_{n\to\infty} a_n = \infty$.

■ **Example 2.5.1** Given a real number α, define

$$a_n = \frac{\alpha^n}{n!},\ n \in \mathbb{N}.$$

When $\alpha = 0$, it is obvious that $\lim_{n\to\infty} a_n = 0$. Suppose $\alpha > 0$. Then

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\alpha}{n+1} = 0 < 1.$$

Thus, $\lim_{n\to\infty} a_n = 0$. In the general case, we can also show that $\lim_{n\to\infty} a_n = 0$ by considering $\lim_{n\to\infty} |a_n|$ and using Exercise 2.1.3.

Exercises

All sequences in this set of exercises are assumed to be in $\mathbb{R}$.

2.5.1 Find $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ for each sequence.

(a) $a_n = (-1)^n$.

(b) $a_n = \sin\left(\frac{n\pi}{2}\right)$.

(c) $a_n = \frac{1+(-1)^n}{n}$.

(d) $a_n = n\sin\left(\frac{n\pi}{2}\right)$.

2.5.2 For a sequence $\{a_n\}$, prove that:

(a) $\liminf_{n\to\infty} a_n = \infty$ if and only if $\lim_{n\to\infty} a_n = \infty$.

(b) $\limsup_{n\to\infty} a_n = -\infty$ if and only if $\lim_{n\to\infty} a_n = -\infty$.

2.5.3 Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences. Prove that:

(a) $\sup_{k\geq n}(a_n + b_n) \leq \sup_{k\geq n} a_k + \sup_{k\geq n} b_k$.

(b) $\inf_{k\geq n}(a_n + b_n) \geq \inf_{k\geq n} a_k + \inf_{k\geq n} b_k$.

2.5.4 ► Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences.

(a) Prove that $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.

(b) Prove that $\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$.

(c) Find two counterexamples to show that the equalities may not hold in part (a) and part (b).

Is the conclusion still true in each of parts (a) and (b) if the sequences involved are not necessarily bounded?

2.5.5 Let $\{a_n\}$ be a convergent sequence and let $\{b_n\}$ be an arbitrary sequence. Prove that

(a) $\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n = \lim_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.

(b) $\liminf_{n\to\infty}(a_n + b_n) = \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n = \lim_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$.

2.6 OPEN SETS, CLOSED SETS, COMPACT SETS, AND LIMIT POINTS

The *open ball* in $\mathbb{R}$ with center $a \in \mathbb{R}$ and radius $\delta > 0$ is the set

$$B(a;\delta) = (a-\delta, a+\delta).$$

Definition 2.6.1 A subset A of $\mathbb{R}$ is said to be *open* if for each $a \in A$, there exists $\delta > 0$ such that

$$B(a;\delta) \subset A.$$

■ **Example 2.6.1** (1) Any open interval $A = (c,d)$ is open. Indeed, for each $a \in A$, one has $c < a < d$. Let

$$\delta = \min\{a-c, d-a\}.$$

Then

$$B(a;\delta) = (a-\delta, a+\delta) \subset A.$$

Therefore, A is open.

(2) The sets $A = (-\infty, c)$ and $B = (c, \infty)$ are open, but the set $C = [c, \infty)$ is not open. The reader can easily verify that A and B are open. Let us show that C is not open. Assume by contradiction that C is open. Then, for the element $c \in C$, there exists $\delta > 0$ such that

$$B(c;\delta) = (c-\delta, c+\delta) \subset C.$$

However, this is a contradiction because $c - \delta/2 \in B(c;\delta)$, but $c - \delta/2 \notin C$.

Theorem 2.6.1 The following hold:

(a) The subsets $\emptyset$ and $\mathbb{R}$ are open.
(b) The union of any collection of open subsets of $\mathbb{R}$ is open.
(c) The intersection of a finite number of open subsets of $\mathbb{R}$ is open.

Proof: The proof of (a) is straightforward.

(b) Suppose $\{G_\alpha : \alpha \in I\}$ is an arbitrary collection of open subsets of $\mathbb{R}$. That means G_α is open for every $\alpha \in I$. Let us show that the set

$$G = \bigcup_{\alpha \in I} G_\alpha$$

is open. Take any $a \in G$. Then there exists $\alpha_0 \in I$ such that

$$a \in G_{\alpha_0}.$$

Since G_{α_0} is open, there exists $\delta > 0$ such that

$$B(a;\delta) \subset G_{\alpha_0}.$$

This implies

$$B(a;\delta) \subset G$$

because $G_{\alpha_0} \subset G$. Thus, G is open.

(c) Suppose $G_i, i = 1, \ldots, n$, are open subsets of $\mathbb{R}$. Let us show that the set

$$G = \bigcap_{i=1}^{n} G_i$$

is also open. Take any $a \in G$. Then $a \in G_i$ for $i = 1, \ldots, n$. Since each G_i is open, there exists $\delta_i > 0$ such that

$$B(a; \delta_i) \subset G_i.$$

Let $\delta = \min\{\delta_i : i = 1, \ldots, n\}$. Then $\delta > 0$ and

$$B(a; \delta) \subset G.$$

Thus, G is open. $\square$

Definition 2.6.2 A subset S of $\mathbb{R}$ is called *closed* if its complement, $S^c = \mathbb{R} \setminus S$, is open.

■ **Example 2.6.2** The sets $[a,b]$, $(-\infty, a]$, and $[a, \infty)$ are closed. Indeed, $(-\infty, a]^c = (a, \infty)$ and $[a, \infty)^c = (-\infty, a)$ which are open by Example 2.6.1. Since $[a,b]^c = (-\infty, a) \cup (b, \infty)$, $[a,b]^c$ is open by Theorem 2.6.1. Also, single element sets are closed since, say, $\{b\}^c = (-\infty, b) \cup (b, \infty)$.

Theorem 2.6.2 The following hold:

(a) The sets $\emptyset$ and $\mathbb{R}$ are closed.
(b) The intersection of any collection of closed subsets of $\mathbb{R}$ is closed.
(c) The union of a finite number of closed subsets of $\mathbb{R}$ is closed.

Proof: The proofs for these are simple using the De Morgan's law. Let us prove, for instance, (b). Let $\{S_\alpha : \alpha \in I\}$ be a collection of closed sets. We will prove that the set

$$S = \bigcap_{\alpha \in I} S_\alpha$$

is also closed. We have

$$S^c = \left(\bigcap_{\alpha \in I} S_\alpha \right)^c = \bigcup_{\alpha \in I} S_\alpha^c.$$

Thus, S^c is open because it is a union of opens sets in $\mathbb{R}$ (Theorem 2.6.1(b)). Therefore, S is closed. $\square$

■ **Example 2.6.3** It follows from part (c) and Example 2.6.2 that any finite set is closed.

Theorem 2.6.3 A subset A of $\mathbb{R}$ is closed if and only if for any sequence $\{a_n\}$ in A that converges to a point $a \in \mathbb{R}$, it follows that $a \in A$.

Proof: Suppose A is a closed subset of $\mathbb{R}$ and $\{a_n\}$ is a sequence in A that converges to a. Suppose by contradiction that $a \notin A$. Since A is closed, there exists $\varepsilon > 0$ such that $B(a; \varepsilon) = (a - \varepsilon, a + \varepsilon) \subset A^c$. Since $\{a_n\}$ converges to a, there exists $N \in \mathbb{N}$ such that

$$a - \varepsilon < a_N < a + \varepsilon.$$

This implies $a_N \in A^c$, a contradiction.

Let us now prove the converse. Suppose by contradiction that A is not closed. Then A^c is not open. Since A^c is not open, there exists $a \in A^c$ such that for any $\varepsilon > 0$, one has $B(a;\varepsilon) \cap A \neq \emptyset$. In particular, for such an a and for each $n \in \mathbb{N}$, there exists $a_n \in B(a;\frac{1}{n}) \cap A$. It is clear that the sequence $\{a_n\}$ is in A and it is convergent to a (because $|a_n - a| < \frac{1}{n}$, for all $n \in \mathbb{N}$). This is a contradiction since $a \notin A$. Therefore, A is closed. $\square$

Theorem 2.6.4 If A is a nonempty subset of $\mathbb{R}$ that is closed and bounded above, then $\max A$ exists. Similarly, if A is a nonempty subset of $\mathbb{R}$ that is closed and bounded below, then $\min A$ exists

Proof: Let A be a nonempty closed set that is bounded above. Then $\sup A$ exists. Let $m = \sup A$. To complete the proof, we will show that $m \in A$. Assume by contradiction that $m \notin A$. Then $m \in A^c$, which is an open set. So there exists $\delta > 0$ such that

$$(m-\delta, m+\delta) \subset A^c.$$

This means there exists no $a \in A$ with

$$m - \delta < a \leq m.$$

This contradicts the fact that m is the least upper bound of A (see Proposition 1.5.1). Therefore, $\max A$ exists. $\square$

Definition 2.6.3 A subset A of $\mathbb{R}$ is called *compact* if for every sequence $\{a_n\}$ in A, there exists a subsequence $\{a_{n_k}\}$ that converges to a point $a \in A$.[1]

■ **Example 2.6.4** Let $a, b \in \mathbb{R}$, $a \leq b$. We show that the set $A = [a,b]$ is compact. Let $\{a_n\}$ be a sequence in A. Since $a \leq a_n \leq b$ for all n, then the sequence is bounded. By the Bolzano-Weierstrass theorem (Theorem 2.4.1), we can obtain a convergent subsequence $\{a_{n_k}\}$. Say, $\lim_{k\to\infty} a_{n_k} = s$. We now must show that $s \in A$. Since $a \leq a_{n_k} \leq b$ for all k, it follows from Theorem 2.1.5, that $a \leq s \leq b$ and, hence, $s \in A$ as desired. We conclude that A is compact.

Theorem 2.6.5 A subset A of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Proof: Suppose A is a compact subset of $\mathbb{R}$. Let us first show that A is bounded. Suppose, by contradiction, that A is not bounded. Then for every $n \in \mathbb{N}$, there exists $a_n \in A$ such that

$$|a_n| \geq n.$$

Since A is compact, there exists a subsequence $\{a_{n_k}\}$ that converges to some $a \in A$. Then

$$|a_{n_k}| \geq n_k \geq k \quad \text{for all } k.$$

Therefore, $\lim_{k\to\infty} |a_{n_k}| = \infty$. This is a contradiction because $\{|a_{n_k}|\}$ converges to $|a|$. Thus A is bounded.

Let us now show that A is closed. Let $\{a_n\}$ be a sequence in A that converges to a point $a \in \mathbb{R}$. By the definition of compactness, $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ that converges to $b \in A$. Then $a = b \in A$ and, hence, A is closed by Theorem 2.6.3.

For the converse, suppose A is closed and bounded and let $\{a_n\}$ be a sequence in A. Since A is bounded, the sequence is bounded and, by the Bolzano-Weierstrass theorem (Theorem 2.4.1), it

[1] This definition of compactness is more commonly referred to as *sequential compactness*.

has a convergent subsequence, $\{a_{n_k}\}$. Say, $\lim_{k\to\infty} a_{n_k} = a$. It now follows from Theorem 2.6.3 that $a \in A$. This shows that A is compact as desired. □

Definition 2.6.4 (cluster/limit/accumulation point). Let A be a subset of $\mathbb{R}$. A point $a \in \mathbb{R}$ (not necessarily in A) is called a *limit point* of A if for any $\delta > 0$, the open ball $B(a;\delta)$ contains an infinite number of points of A.

A point $a \in A$ which is not an accumulation point of A is called an *isolated point of A*.

■ **Example 2.6.5** (1) Let $A = [0,1)$. Then $a = 0$ is a limit point of A and $b = 1$ is also a limit point of A. In fact, any point of the interval $[0,1]$ is a limit point of A. The set $[0,1)$ has no isolated points.

(2) Let $A = \mathbb{Z}$. Then A does not have any limit points. Every element of $\mathbb{Z}$ is an isolated point of $\mathbb{Z}$.

(3) Let $A = \{1/n : n \in \mathbb{N}\}$. Then $a = 0$ is the only limit point of A. All elements of A are isolated points.

■ **Example 2.6.6** If G is an open subset of $\mathbb{R}$ then every point of G is a limit point of G. In fact, more is true. If G is open and $a \in G$, then a is a limit point of $G \setminus \{a\}$. Indeed, let $\delta > 0$ be such that $B(a;\delta) \subset G$. Then $(G \setminus \{a\}) \cap B(a;\delta) = (a-\delta, a) \cup (a, a+\delta)$ and, thus $B(a;\delta)$ contains an infinite number of points of $G \setminus \{a\}$.

The following theorem is a variation of the Bolzano-Weierstrass theorem.

Theorem 2.6.6 Any infinite bounded subset of $\mathbb{R}$ has at least one limit point.

Proof: Let A be an infinite subset of $\mathbb{R}$ and let $\{a_n\}$ be a sequence of A such that

$$a_m \neq a_n \text{ for } m \neq n$$

(see Theorem 1.2.7). Since $\{a_n\}$ is bounded, by the Bolzano-Weierstrass theorem (Theorem 2.4.1), it has a convergent subsequence $\{a_{n_k}\}$. Set $b = \lim_{k\to\infty} a_{n_k}$. Given $\delta > 0$, there exists $K \in \mathbb{N}$ such that $a_{n_k} \in B(b;\delta)$ for $k \geq K$. Since the set $\{a_{n_k} : k \geq K\}$ is infinite, it follows that b is a limit point of A. □

The following definitions and results provide the framework for discussing convergence within subsets of $\mathbb{R}$.

Definition 2.6.5 Let D be a subset of $\mathbb{R}$. We say that a subset V of D is *open in D* if for every $a \in V$, there exists $\delta > 0$ such that

$$B(a;\delta) \cap D \subset V.$$

Theorem 2.6.7 Let D be a subset of $\mathbb{R}$. A subset V of D is open in D if and only if there exists an open subset G of $\mathbb{R}$ such that

$$V = D \cap G.$$

Proof: Suppose V is open in D. By definition, for every $a \in V$, there exists $\delta_a > 0$ such that

$$B(a;\delta_a) \cap D \subset V.$$

Define

$$G = \cup_{a \in V} B(a;\delta_a)$$

Then G is a union of open subsets of $\mathbb{R}$, so G is open. Moreover,

$$V \subset G \cap D = \cup_{a \in V}[B(a;\delta_a) \cap D] \subset V.$$

Therefore, $V = G \cap D$.

Let us now prove the converse. Suppose $V = G \cap D$, where G is an open set. For any $a \in V$, we have $a \in G$, so there exists $\delta > 0$ such that

$$B(a;\delta) \subset G.$$

It follows that

$$B(a;\delta) \cap D \subset G \cap D = V.$$

The proof is now complete. □

■ **Example 2.6.7** Let $D = [0,1)$ and $V = [0,\frac{1}{2})$. We can write $V = D \cap (-1,\frac{1}{2})$. Since $(-1,\frac{1}{2})$ is open in $\mathbb{R}$, we conclude from Theorem 2.6.7 that V is open in D. Notice that V itself is not an open subset of $\mathbb{R}$.

The following theorem is now a direct consequence of Theorems 2.6.7 and 2.6.1.

Theorem 2.6.8 Let D be a subset of $\mathbb{R}$. The following hold:

(a) The subsets $\emptyset$ and D are open in D.
(b) The union of any collection of open sets in D is open in D.
(c) The intersection of a finite number of open sets in D is open in D.

Definition 2.6.6 Let D be a subset of $\mathbb{R}$. We say that a subset A of D is *closed in D* if $D \setminus A$ is open in D.

Theorem 2.6.9 Let D be a subset of $\mathbb{R}$. A subset K of D is closed in D if and only if there exists a closed subset F of $\mathbb{R}$ such that

$$K = D \cap F.$$

Proof: Suppose K is a closed set in D. Then $D \setminus K$ is open in D. By Theorem 2.6.7, there exists an open set G such that

$$D \setminus K = D \cap G.$$

It follows that

$$K = D \setminus (D \setminus K) = D \setminus (D \cap G) = D \setminus G = D \cap G^c.$$

Let $F = G^c$. Then F is a closed subset of $\mathbb{R}$ and $K = D \cap F$.

Conversely, suppose that there exists a closed subset F of $\mathbb{R}$ such that $K = D \cap F$. Then

$$D \setminus K = D \setminus (D \cap F) = D \setminus F = D \cap F^c.$$

Since F^c is an open subset of $\mathbb{R}$, applying Theorem 2.6.7 again, one has that $D \setminus K$ is open in D. Therefore, K is closed in D by definition. □

■ **Example 2.6.8** Let $D = [0,1)$ and $K = [\frac{1}{2},1)$. We can write $K = D \cap [\frac{1}{2},2]$. Since $[\frac{1}{2},2]$ is closed in $\mathbb{R}$, we conclude from Theorem 2.6.9 that K is closed in D. Notice that K itself is not a closed subset of $\mathbb{R}$.

Corollary 2.6.10 Let D be a subset of $\mathbb{R}$. A subset K of D is closed in D if and only if for every sequence $\{x_k\}$ in K that converges to a point $\bar{x} \in D$ it follows that $\bar{x} \in K$.

Proof: Let D be a subset of $\mathbb{R}$. Suppose K is closed in D. By Theorem 2.6.9, there exists a closed subset F of $\mathbb{R}$ such that

$$K = D \cap F.$$

Let $\{x_k\}$ be a sequence in K that converges to a point $\bar{x} \in D$. Since $\{x_k\}$ is also a sequence in F and F is a closed subset of $\mathbb{R}$, $\bar{x} \in F$. Thus, $\bar{x} \in D \cap F = K$.

Let us prove the converse. Suppose by contradiction that K is not closed in D or $D \setminus K$ is not open in D. Then there exists $\bar{x} \in D \setminus K$ such that for every $\delta > 0$, one has

$$B(\bar{x};\delta) \cap D \not\subseteq D \setminus K.$$

In particular, for every $k \in \mathbb{N}$,

$$B\left(\bar{x};\frac{1}{k}\right) \cap D \not\subseteq D \setminus K.$$

For each $k \in \mathbb{N}$, choose $x_k \in B(\bar{x};\frac{1}{k}) \cap D$ such that $x_k \notin D \setminus K$. Then $\{x_k\}$ is a sequence in K and, moreover, $\{x_k\}$ converges to $\bar{x} \in D$. Then $\bar{x} \in K$. This is a contradiction. We conclude that K is closed in D. □

The following theorem is a direct consequence of Theorems 2.6.9 and 2.6.2.

Theorem 2.6.11 Let D be a subset of $\mathbb{R}$. The following hold:

(a) The subsets $\emptyset$ and D are closed in D.

(b) The intersection of any collection of closed sets in D is closed in D.

(c) The union of a finite number of closed sets in D is closed in D.

■ **Example 2.6.9** Consider the set $D = [0,1)$ and the subset $A = [\frac{1}{2},1)$. Clearly, A is bounded. We showed in Example 2.6.8 that A is closed in D. However, A is not compact. We show this by finding a sequence $\{a_n\}$ in A for which no subsequence converges to a point in A.

Indeed, consider the sequence $a_n = 1 - \frac{1}{2n}$ for $n \in \mathbb{N}$. Then $a_n \in A$ for all n. Moreover, $\{a_n\}$ converges to 1 and, hence, every subsequence also converges to 1. Since $1 \notin A$, it follows that A is not compact.

Exercises

2.6.1 Prove that a subset A of $\mathbb{R}$ is open if and only if for any $x \in A$, there exists $n \in \mathbb{N}$ such that $(x - 1/n, x + 1/n) \subset A$.

2.6.2 Prove that the interval $[0,1)$ is neither open nor closed.

2.6.3 ▶ Prove that if A and B are compact subsets of $\mathbb{R}$, then $A \cup B$ is a compact set.

2.6.4 Prove that the intersection of any collection of compact subsets of $\mathbb{R}$ is compact.

2.6.5 Find all limit points and all isolated points of each of the following sets:

(a) $A = (0,1)$.
(b) $B = [0,1)$.
(c) $C = \mathbb{Q}$.
(d) $D = \{m + 1/n \ : \ m,n \in \mathbb{N}\}$.

2.6.6 Let $D = [0,\infty)$. Classify each subset of D below as open in D, closed in D, neither or both. Justify your answers.

(a) $A = (0,1)$.
(b) $B = \mathbb{N}$.
(c) $C = \mathbb{Q} \cap D$.
(d) $D = (-1,1]$.
(e) $E = (-2,\infty)$.

3. LIMITS AND CONTINUITY

In this chapter, we extend our analysis of limit processes to functions and give the precise definition of continuous function. We derive rigorously two fundamental theorems about continuous functions: the extreme value theorem and the intermediate value theorem.

3.1 LIMITS OF FUNCTIONS

Definition 3.1.1 Let $f: D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. We say that f *has a limit at* $\bar{x}$ if there exists a real number ℓ such that for every $\varepsilon > 0$, there exists $\delta > 0$ with

$$|f(x) - \ell| < \varepsilon$$

for all $x \in D$ for which $0 < |x - \bar{x}| < \delta$. In this case, we write

$$\lim_{x \to \bar{x}} f(x) = \ell.$$

Remark 3.1.1 Note that the limit point $\bar{x}$ in the definition of limit may or may not be an element of the domain D. In any case, the inequality $|f(x) - \ell| < \varepsilon$ need only be satisfied by elements of D.

■ **Example 3.1.1** Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 5x - 7$. We prove that $\lim_{x \to 2} f(x) = 3$. Let $\varepsilon > 0$. First note that $|f(x) - 2| = |5x - 7 - 3| = |5x - 10| = 5|x - 2|$. This suggests the choice $\delta = \varepsilon/5$. Then, if $|x - 2| < \delta$ we have

$$|f(x) - 2| = 5|x - 2| < 5\delta = \varepsilon.$$

■ **Example 3.1.2** Let $f: [0,1) \to \mathbb{R}$ be given by $f(x) = x^2 + x$. Let $\bar{x} = 1$ and $\ell = 2$. First note that $|f(x) - \ell| = |x^2 + x - 2| = |x - 1||x + 2|$ and for $x \in [0,1)$, $|x + 2| \le |x| + 2 \le 3$. Now, given $\varepsilon > 0$, choose $\delta = \varepsilon/3$. Then, if $|x - 1| < \delta$ and $x \in [0,1)$, we have

$$|f(x) - \ell| = |x^2 + x - 2| = |x - 1||x + 2| < 3\delta = 3\frac{\varepsilon}{3} = \varepsilon.$$

This shows that $\lim_{x \to 1} f(x) = 2$.

■ **Example 3.1.3** Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. We show that $\lim_{x\to 2} f(x) = 4$. First note that $|f(x)-4| = |x^2-4| = |(x-2)(x+2)| = |x-2||x+2|$. Since the domain is all of $\mathbb{R}$ the expression $|x+2|$ is not bounded and we cannot proceed as in Example 3.1.2. However, we are interested only in values of x close to 2 and, thus, we impose the condition $\delta \le 1$. If $|x-2| < 1$, then $-1 < x-2 < 1$ and, so, $1 < x < 3$. It follows, for such x, that $|x| < 3$ and, hence $|x|+2 < 5$.

Now, given $\varepsilon > 0$ we choose $\delta = \min\{1, \frac{\varepsilon}{5}\}$. Then, whenver $|x-2| < \delta$ we get

$$|f(x)-4| = |x-2|x+2| \le |x-2|(|x|+2) < \delta 5 \le \varepsilon.$$

■ **Example 3.1.4** Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \dfrac{3x-5}{x^2+3}$. We prove that $\lim_{x\to 1} f(x) = -\dfrac{1}{2}$. First we look at the expression $|f(x) - (-\frac{1}{2})|$ and try to identify a factor $|x-1|$ (because here $\bar{x} = 1$).

$$\left| f(x) - \left(-\frac{1}{2}\right) \right| = \left| \frac{3x-5}{x^2+3} + \frac{1}{2} \right| = \left| \frac{6x-10+x^2+3}{x^2+3} \right| = \frac{|x-1||x+7|}{2(x^2+3)} \le \frac{1}{6}|x-1||x+7|.$$

Proceeding as in the previous example, if $|x-1| < 1$ we get $-1 < x-1 < 1$ and, so, $0 < x < 2$. Thus $|x| < 2$ and $|x+7| \le |x|+2 < 9$.

Now, given $\varepsilon > 0$, we choose $\delta = \min\{1, \frac{2}{3}\varepsilon\}$. It follows that if $|x-1| < \delta$ we get

$$\left| f(x) - (-\frac{1}{2}) \right| \le \frac{|x+7|}{6}|x-1| < \frac{9}{6}\delta \le \varepsilon.$$

The following theorem will let us apply our earlier results on limits of sequences to obtain new results on limits of functions.

Theorem 3.1.2 — Sequential Characterization of Limits. Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Then

$$\lim_{x\to\bar{x}} f(x) = \ell \tag{3.1}$$

if and only if

$$\lim_{n\to\infty} f(x_n) = \ell \tag{3.2}$$

for every sequence $\{x_n\}$ in D such that $x_n \ne \bar{x}$ for every n and $\{x_n\}$ converges to $\bar{x}$.

Proof: Suppose (3.1) holds. Let $\{x_n\}$ be a sequence in D with $x_n \ne \bar{x}$ for every n and such that $\{x_n\}$ converges to $\bar{x}$. Given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x)-\ell| < \varepsilon$ whenever $x \in D$ and $0 < |x-\bar{x}| < \delta$. Then there exists $N \in \mathbb{N}$ with $0 < |x_n - \bar{x}| < \delta$ for all $n \ge N$. For such n, we have

$$|f(x_n) - \ell| < \varepsilon.$$

This implies (3.2).

Conversely, suppose (3.1) is false. Then there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$, there exists $x \in D$ with $0 < |x-\bar{x}| < \delta$ and $|f(x)-\ell| \ge \varepsilon_0$. Thus, for every $n \in \mathbb{N}$, there exists $x_n \in D$ with $0 < |x_n - \bar{x}| < \dfrac{1}{n}$ and $|f(x_n) - \ell| \ge \varepsilon_0$. By the squeeze theorem (Theorem 2.1.6), the sequence $\{x_n\}$ converges to $\bar{x}$. Moreover, $x_n \ne \bar{x}$ for every n. This shows that (3.2) is false. It follows that (3.2) implies (3.1) and the proof is complete. □

Corollary 3.1.3 Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. If f has a limit at $\bar{x}$, then this limit is unique.

Proof: Suppose by contradiction that f has two different limits ℓ_1 and ℓ_2. Let $\{x_n\}$ be a sequence in $D \setminus \{\bar{x}\}$ that converges to $\bar{x}$. By Theorem 3.1.2, the sequence $\{f(x_n)\}$ converges to two different limits ℓ_1 and ℓ_2. This is a contradiction to Theorem 2.1.3. $\square$

The following corollary follows directly from Theorem 3.1.2.

Corollary 3.1.4 Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Then f does not have a limit at $\bar{x}$ if and only if there exists a sequence $\{x_n\}$ in D such that $x_n \neq \bar{x}$ for every n, $\{x_n\}$ converges to $\bar{x}$, and $\{f(x_n)\}$ does not converge.

■ **Example 3.1.5** Consider the Dirichlet function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}; \\ 0, & \text{if } x \in \mathbb{Q}^c. \end{cases}$$

Then $\lim_{x\to\bar{x}} f(x)$ does not exist for any $\bar{x} \in \mathbb{R}$. Indeed, fix $\bar{x} \in \mathbb{R}$ and choose two sequences $\{r_n\}$, $\{s_n\}$ converging to $\bar{x}$ such that $r_n \in \mathbb{Q}$ and $s_n \notin \mathbb{Q}$ for all $n \in \mathbb{N}$. Define a new sequence $\{x_n\}$ by

$$x_n = \begin{cases} r_k, & \text{if } n = 2k; \\ s_k, & \text{if } n = 2k-1. \end{cases}$$

It is clear that $\{x_n\}$ converges to $\bar{x}$. Moreover, since $\{f(r_n)\}$ converges to 1 and $\{f(s_n)\}$ converges to 0, Theorem 2.1.9 implies that the sequence $\{f(x_n)\}$ does not converge. It follows from the sequential characterization of limits that $\lim_{x\to\bar{x}} f(x)$ does not exist.

Theorem 3.1.5 Let $f, g\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Suppose that

$$\lim_{x\to\bar{x}} f(x) = \ell_1, \ \lim_{x\to\bar{x}} g(x) = \ell_2,$$

and that there exists $\delta > 0$ such that

$$f(x) \leq g(x) \text{ for all } x \in B(\bar{x};\delta) \cap D, x \neq \bar{x}.$$

Then $\ell_1 \leq \ell_2$.

Proof: Let $\{x_n\}$ be a sequence in $B(\bar{x};\delta) \cap D = (\bar{x}-\delta, \bar{x}+\delta) \cap D$ that converges to $\bar{x}$ and $x_n \neq \bar{x}$ for all n. By Theorem 3.1.2,

$$\lim_{n\to\infty} f(x_n) = \ell_1 \text{ and } \lim_{n\to\infty} g(x_n) = \ell_2.$$

Since $f(x_n) \leq g(x_n)$ for all $n \in \mathbb{N}$, applying Theorem 2.1.5, we obtain $\ell_1 \leq \ell_2$. $\square$

Theorem 3.1.6 Let $f, g\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Suppose

$$\lim_{x\to\bar{x}} f(x) = \ell_1, \ \lim_{x\to\bar{x}} g(x) = \ell_2,$$

and $\ell_1 < \ell_2$. Then there exists $\delta > 0$ such that

$$f(x) < g(x) \text{ for all } x \in B(\bar{x};\delta) \cap D, x \neq \bar{x}.$$

Proof: Choose $\varepsilon > 0$ such that $\ell_1 + \varepsilon < \ell_2 - \varepsilon$ (equivalently, such that $\varepsilon < \frac{\ell_2 - \ell_1}{2}$). Then there exists $\delta > 0$ such that

$$\ell_1 - \varepsilon < f(x) < \ell_1 + \varepsilon \text{ and } \ell_2 - \varepsilon < g(x) < \ell_2 + \varepsilon$$

for all $x \in B(\bar{x}; \delta) \cap D, x \neq \bar{x}$. Thus,

$$f(x) < \ell_1 + \varepsilon < \ell_2 - \varepsilon < g(x) \text{ for all } x \in B(\bar{x}; \delta) \cap D, x \neq \bar{x}.$$

The proof is now complete. □

Theorem 3.1.7 Let $f, g, h \colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Suppose there exists $\delta > 0$ such that $f(x) \leq g(x) \leq h(x)$ for all $x \in B(\bar{x}; \delta) \cap D, x \neq \bar{x}$. If $\lim_{x \to \bar{x}} f(x) = \lim_{x \to \bar{x}} h(x) = \ell$, then $\lim_{x \to \bar{x}} g(x) = \ell$.

Proof: The proof is straightforward using Theorem 2.1.6 and Theorem 3.1.2. □

Remark 3.1.8 We will adopt the following convention. When we write $\lim_{x \to \bar{x}} f(x)$ without specifying the domain D of f we will assume that D is the largest subset of $\mathbb{R}$ such that if $x \in D$, then $f(x)$ results in a real number. For example, in

$$\lim_{x \to 2} \frac{1}{x+3}$$

we assume $D = \mathbb{R} \setminus \{-3\}$ and in

$$\lim_{x \to 1} \sqrt{x}$$

we assume $D = [0, \infty)$.

Exercises

3.1.1 Use the definition of limit to prove that

(a) $\lim_{x \to 2} 3x - 7 = -1$.
(b) $\lim_{x \to 3} (x^2 + 1) = 10$.
(c) $\lim_{x \to 1} \dfrac{x+3}{x+1} = 2$.
(d) $\lim_{x \to 0} \sqrt{x} = 0$.
(e) $\lim_{x \to 2} x^3 = 8$.

3.1.2 Prove that the following limits do not exist.

(a) $\lim_{x \to 0} \dfrac{x}{|x|}$.
(b) $\lim_{x \to 0} \cos(1/x)$.

3.1.3 Let $f \colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Prove that if $\lim_{x \to \bar{x}} f(x) = \ell$, then

$$\lim_{x \to \bar{x}} |f(x)| = |\ell|.$$

Give an example to show that the converse is not true in general.

3.1.4 Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Suppose $f(x) \geq 0$ for all $x \in D$. Prove that if $\lim_{x\to\bar{x}} f(x) = \ell$, then

$$\lim_{x\to\bar{x}} \sqrt{f(x)} = \sqrt{\ell}.$$

3.1.5 Find $\lim_{x\to 0} x\sin(1/x)$.

3.1.6 ▶ Let f be the function given by

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \cap [0,1]; \\ 1-x, & \text{if } x \in \mathbb{Q}^c \cap [0,1]. \end{cases}$$

Determine which of the following limits exist. For those that exist find their values.

(a) $\lim_{x\to 1/2} f(x)$.
(b) $\lim_{x\to 0} f(x)$.
(c) $\lim_{x\to 1} f(x)$.

3.2 LIMIT THEOREMS

Here we state and prove various theorems that facilitate the computation of general limits.

Definition 3.2.1 Let $f,g\colon D \to \mathbb{R}$ and let c be a constant. The functions $f+g$, fg, and cf are respectively defined as functions from D to $\mathbb{R}$ by

$$(f+g)(x) = f(x) + g(x),$$
$$(fg)(x) = f(x)g(x),$$
$$(cf)(x) = cf(x)$$

for $x \in D$. Let $\widetilde{D} = \{x \in D : g(x) \neq 0\}$. The function $\dfrac{f}{g}$ is defined as a function from $\widetilde{D}$ to $\mathbb{R}$ by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

for $x \in \widetilde{D}$.

Theorem 3.2.1 Let $f,g\colon D \to \mathbb{R}$ and let $c \in \mathbb{R}$. Suppose $\bar{x}$ is a limit point of D and

$$\lim_{x\to\bar{x}} f(x) = \ell,\ \lim_{x\to\bar{x}} g(x) = m.$$

Then

(a) $\lim_{x\to\bar{x}}(f+g)(x) = \ell + m$,
(b) $\lim_{x\to\bar{x}}(fg)(x) = \ell m$,
(c) $\lim_{x\to\bar{x}}(cf)(x) = c\ell$,
(d) $\lim_{x\to\bar{x}} \left(\dfrac{f}{g}\right)(x) = \dfrac{\ell}{m}$ provided that $m \neq 0$.

Proof: Let us first prove (a). Let $\{x_n\}$ be a sequence in D that converges to $\bar{x}$ and $x_n \neq \bar{x}$ for every n. By Theorem 3.1.2,

$$\lim_{n\to\infty} f(x_n) = \ell \text{ and } \lim_{n\to\infty} g(x_n) = m.$$

It follows from Theorem 2.2.1 that

$$\lim_{n\to\infty}(f(x_n)+g(x_n)) = \ell + m.$$

Applying Theorem 3.1.2 again, we get $\lim_{x\to\bar{x}}(f+g)(x) = \ell+m$. The proofs of (b) and (c) are similar.

Let us now show that if $m \neq 0$, then $\bar{x}$ is a limit point of $\widetilde{D}$. Since $\bar{x}$ is a limit point of D, there is a sequence $\{u_k\}$ in D converging to $\bar{x}$ such that $u_k \neq \bar{x}$ for every k. Since $m \neq 0$, it follows from an easy application of Theorem 3.1.6 that there exists $\delta > 0$ with

$$g(x) \neq 0 \text{ whenever } 0 < |x-\bar{x}| < \delta, x \in D.$$

This implies

$$x \in \widetilde{D} \text{ whenever } 0 < |x-\bar{x}| < \delta, x \in D.$$

Then $u_k \in \widetilde{D}$ for all k sufficiently large, and hence $\bar{x}$ is a limit point of $\widetilde{D}$. The rest of the proof of (d) can be completed easily following the proof of (a). □

■ **Example 3.2.1** Consider $f\colon \mathbb{R}\setminus\{-7\} \to \mathbb{R}$ given by $f(x) = \dfrac{x^2+2x-3}{x+7}$. Then, combining all parts of Theorem 3.2.1, we get

$$\lim_{x\to-2} f(x) = \frac{\lim_{x\to-2}(x^2+2x-3)}{\lim_{x\to-2}(x+7)} = \frac{\lim_{x\to-2}x^2+\lim_{x\to-2}2x-\lim_{x\to-2}3}{\lim_{x\to-2}x+\lim_{x\to-2}7}$$
$$= \frac{(\lim_{x\to-2}x)^2+2\lim_{x\to-2}x-\lim_{x\to-2}3}{\lim_{x\to-2}x+\lim_{x\to-2}7} = \frac{(-2)^2+2(-2)-3}{-2+7} = -\frac{3}{5}.$$

■ **Example 3.2.2** We proceed in the same way to compute the following limit.

$$\lim_{x\to 0}\frac{1+(2x-1)^2}{x^2+7} = \frac{\lim_{x\to 0}1+\lim_{x\to 0}(2x-1)^2}{\lim_{x\to 0}x^2+\lim_{x\to 0}7} = \frac{1+1}{0+7} = \frac{2}{7}.$$

■ **Example 3.2.3** We now consider

$$\lim_{x\to -1}\frac{x^2+6x+5}{x+1}.$$

Since the limit of the denominator is 0 we cannot apply directly part (d) of Theorem 3.2.1. Instead, we first simplify the expression keeping in mind that in the definition of limit we never need to evaluate the expression at the limit point itself. In this case, this means we may assume that $x \neq -1$. For any such x we have

$$\frac{x^2+6x+5}{x+1} = \frac{(x+1)(x+5)}{x+1} = x+5.$$

Therefore,

$$\lim_{x\to -1}\frac{x^2+6x+5}{x+1} = \lim_{x\to -1} x+5 = 4.$$

Theorem 3.2.2 (Cauchy's criterion) Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Then f has a limit at $\bar{x}$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(r) - f(s)| < \varepsilon \text{ whenever } r, s \in D \text{ and } 0 < |r - \bar{x}| < \delta, 0 < |s - \bar{x}| < \delta. \tag{3.3}$$

Proof: Suppose $\lim_{x \to \bar{x}} f(x) = \ell$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - \ell| < \frac{\varepsilon}{2} \text{ whenever } x \in D \text{ and } 0 < |x - \bar{x}| < \delta.$$

Thus, for $r, s \in D$ with $0 < |r - \bar{x}| < \delta$ and $0 < |s - \bar{x}| < \delta$, we have

$$|f(r) - f(s)| \leq |f(r) - \ell| + |\ell - f(s)| < \varepsilon.$$

Let us prove the converse. Fix a sequence $\{u_n\}$ in D such with $\lim_{n\to\infty} u_n = \bar{x}$ and $u_n \neq \bar{x}$ for every n. Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(r) - f(s)| < \varepsilon \text{ whenever } r, s \in D \text{ and } 0 < |r - \bar{x}| < \delta, 0 < |s - \bar{x}| < \delta.$$

Then there exists $N \in \mathbb{N}$ satisfying

$$0 < |u_n - \bar{x}| < \delta \text{ for all } n \geq N.$$

This implies

$$|f(u_n) - f(u_m)| < \varepsilon \text{ for all } m, n \geq N.$$

Thus, $\{f(u_n)\}$ is a Cauchy sequence, and hence there exists $\ell \in \mathbb{R}$ such that

$$\lim_{n\to\infty} f(u_n) = \ell.$$

We now prove that f has limit ℓ at $\bar{x}$ using Theorem 3.1.2. Let $\{x_n\}$ be a sequence in D such that $\lim_{n\to\infty} x_n = \bar{x}$ and $x_n \neq \bar{x}$ for every n. By the previous argument, there exists $\ell' \in \mathbb{R}$ such that

$$\lim_{n\to\infty} f(x_n) = \ell'.$$

Fix any $\varepsilon > 0$ and let $\delta > 0$ satisfy (3.3). There exists $K \in \mathbb{N}$ such that

$$|u_n - \bar{x}| < \delta \text{ and } |x_n - \bar{x}| < \delta$$

for all $n \geq K$. Then $|f(u_n) - f(x_n)| < \varepsilon$ for such n. Letting $n \to \infty$, we have $|\ell - \ell'| \leq \varepsilon$. Thus, $\ell = \ell'$ since ε is arbitrary. It now follows from Theorem 3.1.2 that $\lim_{x\to\bar{x}} f(x) = \ell$. $\square$

The rest of this section discussed some special limits and their properties.

Definition 3.2.2 Let $a \in \mathbb{R}$ and $\delta > 0$. Define

$$B_-(a;\delta) = (a - \delta, a) \text{ and } B_+(a;\delta) = (a, a + \delta).$$

Given a subset A of $\mathbb{R}$, we say that a is a *left limit point of* A if for any $\delta > 0$, $B_-(a;\delta)$ contains an infinite number of elements of A. Similarly, a is called a *right limit point of* A if for any $\delta > 0$, $B_+(a;\delta)$ contains an infinite number of elements of A.

It follows from the definition that a is a limit point of A if and only if it is a left limit point of A or it is a right limit point of A.

Definition 3.2.3 (One-sided limits) Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a left limit point of D. We write

$$\lim_{x \to \bar{x}^-} f(x) = \ell$$

if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - \ell| < \varepsilon \text{ for all } x \in B_-(\bar{x}; \delta).$$

We say that ℓ is the *left-hand limit of f at $\bar{x}$*. The *right-hand limit of f at $\bar{x}$* can be defined in a similar way and is denoted $\lim_{x \to \bar{x}^+} f(x)$.

■ **Example 3.2.4** Consider the function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by

$$f(x) = \frac{|x|}{x}.$$

Let $\bar{x} = 0$. Note first that 0 is a limit point of the set $D = \mathbb{R} \setminus \{0\} \to \mathbb{R}$. Since, for $x > 0$, we have $f(x) = x/x = 1$, we have

$$\lim_{x \to \bar{x}^+} f(x) = \lim_{x \to 0^+} 1 = 1.$$

Similarly, for $x < 0$ we have $f(x) = -x/x = -1$. Therefore,

$$\lim_{x \to \bar{x}^-} f(x) = \lim_{x \to 0^-} -1 = -1.$$

■ **Example 3.2.5** Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x + 4, & \text{if } x < -1; \\ x^2 - 1, & \text{if } x \geq -1. \end{cases} \tag{3.4}$$

We have

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} x^2 - 1 = 0,$$

and

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} x + 4 = 3,$$

The following theorem follows directly from the definition of one-sided limits.

Theorem 3.2.3 Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be both a left limit point of D and a right limit point of D. Then

$$\lim_{x \to \bar{x}} f(x) = \ell$$

if and only if

$$\lim_{x \to \bar{x}^+} f(x) = \ell \text{ and } \lim_{x \to \bar{x}^-} f(x) = \ell.$$

■ **Example 3.2.6** It follows from Example 3.2.4 that $\lim_{x\to 0}\frac{|x|}{x}$ does not exists, since the one-sided limits do not agree.

Definition 3.2.4 (monotonicity) Let $f\colon (a,b)\to\mathbb{R}$.

(1) We say that f is *increasing* on (a,b) if, for all $x_1,x_2\in(a,b)$, $x_1<x_2$ implies $f(x_1)\le f(x_2)$.
(2) We say that f is *decreasing* on (a,b) if, for all $x_1,x_2\in(a,b)$, $x_1<x_2$ implies $f(x_1)\ge f(x_2)$.

If f is increasing or decreasing on (a,b), we say that f is monotone on this interval. Strict monotonicity can be defined similarly using strict inequalities: $f(x_1)<f(x_2)$ in (1) and $f(x_1)>f(x_2)$ in (2).

Theorem 3.2.4 Suppose $f\colon (a,b)\to\mathbb{R}$ is increasing on (a,b) and $\bar{x}\in(a,b)$. Then $\lim_{x\to\bar{x}^-}f(x)$ and $\lim_{x\to\bar{x}^+}f(x)$ exist. Moreover,

$$\sup_{a<x<\bar{x}} f(x)=\lim_{x\to\bar{x}^-}f(x)\le f(\bar{x})\le \lim_{x\to\bar{x}^+}f(x)=\inf_{\bar{x}<x<b}f(x).$$

Proof: Since $f(x)\le f(\bar{x})$ for all $x\in(a,\bar{x})$, the set

$$\{f(x) : x\in(a,\bar{x})\}$$

is nonempty and bounded above. Thus,

$$\ell=\sup_{a<x<\bar{x}} f(x)$$

is a real number. We will show that $\lim_{x\to\bar{x}^-}f(x)=\ell$. For any $\varepsilon>0$, by the definition of the least upper bound, there exists $a<x_1<\bar{x}$ such that

$$\ell-\varepsilon<f(x_1).$$

Let $\delta=\bar{x}-x_1>0$. Using the increasing monotonicity, we get

$$\ell-\varepsilon<f(x_1)\le f(x)\le\ell<\ell+\varepsilon \text{ for all } x\in(x_1,\bar{x})=B_-(\bar{x};\delta).$$

Therefore, $\lim_{x\to\bar{x}^-}f(x)=\ell$. The rest of the proof of the theorem is similar. $\square$

Let

$$B_0(\bar{x};\delta)=B_-(\bar{x};\delta)\cup B_+(\bar{x};\delta)=(\bar{x}-\delta,\bar{x}+\delta)\setminus\{\bar{x}\}.$$

Definition 3.2.5 (infinite limits) Let $f\colon D\to\mathbb{R}$ and let $\bar{x}$ be a limit point of D. We write

$$\lim_{x\to\bar{x}}f(x)=\infty$$

if for every $M\in\mathbb{R}$, there exists $\delta>0$ such that

$$f(x)>M \text{ for all } x\in B_0(\bar{x};\delta)\cap D.$$

Similarly, we write

$$\lim_{x\to\bar{x}}f(x)=-\infty$$

if for every $M\in\mathbb{R}$, there exists $\delta>0$ such that

$$f(x)<M \text{ for all } x\in B_0(\bar{x};\delta)\cap D.$$

Infinite limits of functions have similar properties to those of sequences from Chapter 2 (see Definition 2.3.2 and Theorem 2.3.6).

■ **Example 3.2.7** We show that $\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$ directly from Definition 3.2.5.

Let $M \in \mathbb{R}$. We want to find $\delta > 0$ that will guarantee $\frac{1}{(x-1)^2} > M$ whenever $0 < |x-1| < \delta$. As in the case of finite limits, we work backwards from $\frac{1}{(x-1)^2} > M$ to an inequality for $|x-1|$. To simplify calculations, note that $|M|+1 > M$. Next note that $\frac{1}{(x-1)^2} > |M|+1$, is equivalent to $\sqrt{\frac{1}{|M|+1}} > |x-1|$.

Now, choose δ such that $0 < \delta < \sqrt{\frac{1}{|M|+1}}$. Then, if $0 < |x-1| < \delta$ we have

$$\frac{1}{(x-1)^2} > \frac{1}{\delta^2} > \frac{1}{\frac{1}{|M|+1}} = |M|+1 > M,$$

as desired.

Definition 3.2.6 (limits at infinity) Let $f\colon D \to \mathbb{R}$, where D is not bounded above. We write

$$\lim_{x\to\infty} f(x) = \ell$$

if for every $\varepsilon > 0$, there exists $c \in \mathbb{R}$ such that

$$|f(x)-\ell| < \varepsilon \text{ for all } x > c, x \in D.$$

Let $f\colon D \to \mathbb{R}$, where D is not bounded below. We write

$$\lim_{x\to-\infty} f(x) = \ell$$

if for every $\varepsilon > 0$, there exists $c \in \mathbb{R}$ such that

$$|f(x)-\ell| < \varepsilon \text{ for all } x < c, x \in D.$$

We can also define

$$\lim_{x\to\infty} f(x) = \pm\infty \text{ and } \lim_{x\to-\infty} f(x) = \pm\infty$$

in a similar way.

■ **Example 3.2.8** We prove from the definition that

$$\lim_{x\to-\infty} \frac{3x^2+x}{2x^2+1} = \frac{3}{2}.$$

The approach is similar to that for sequences, with the difference that x need not be an integer.

Let $\varepsilon > 0$. We want to identify c so that

$$\left|\frac{3x^2+x}{2x^2+1} - \frac{3}{2}\right| < \varepsilon, \tag{3.5}$$

for all $x < c$.

Now, $\left|\frac{3x^2+x}{2x^2+1}-\frac{3}{2}\right| = \frac{|2x-3|}{2(2x^2+1)}$. Therefore, simplifying, 3.5 is equivalent to

$$\frac{1}{\varepsilon} < \frac{2(2x^2+1)}{|2x-3|}. \tag{3.6}$$

We first restrict x to be less than 0, so $|2x-3| > 3$. Then, since $\frac{4x^2}{3} < \frac{2(2x^2+1)}{|2x-3|}$, 3.6 will be guaranteed if $1/\varepsilon < 4x^2/3$ or, equivalently $\sqrt{3/(4\varepsilon)} < |x|$. We set $c < \min\{0, -\sqrt{3/(4\varepsilon)}\}$. Then, if $x < c$, we have $\sqrt{3/(4\varepsilon)} < -x = |x|$. Thus, $1/\varepsilon < \frac{2(2x^2+1)}{|2x-3|}$ and, hence,

$$\left|\frac{3x^2+x}{2x^2+1}-\frac{3}{2}\right| = \frac{|2x-3|}{2(2x^2+1)} < \varepsilon.$$

Exercises

3.2.1 Find the following limits:

(a) $\lim_{x\to 2} \frac{3x^2-2x+5}{x-3}$,

(b) $\lim_{x\to -3} \frac{x^2+4x+3}{x^2-9}$

3.2.2 Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ is a limit point of D. Prove that if $\lim_{x\to\bar{x}} f(x)$ exists, then

$$\lim_{x\to\bar{x}} [f(x)]^n = [\lim_{x\to\bar{x}} f(x)]^n, \text{for any } n \in \mathbb{N}.$$

3.2.3 Find the following limits:

(a) $\lim_{x\to 1} \frac{\sqrt{x}-1}{x^2-1}$,

(b) $\lim_{x\to 1} \frac{x^m-1}{x^n-1}$, where $m,n \in \mathbb{N}$,

(c) $\lim_{x\to 1} \frac{\sqrt[n]{x}-1}{\sqrt[m]{x}-1}$, where $m,n \in \mathbb{N}$, $m,n \geq 2$,

(d) $\lim_{x\to 1} \frac{\sqrt{x}-\sqrt[3]{x}}{x-1}$.

3.2.4 Find the following limits:

(a) $\lim_{x\to\infty} (\sqrt[3]{x^3+3x^2} - \sqrt{x^2+1})$.

(b) $\lim_{x\to-\infty} (\sqrt[3]{x^3+3x^2} - \sqrt{x^2+1})$.

3.2.5 ▶ Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Suppose that

$$|f(x)-f(y)| \leq k|x-y| \text{ for all } x,y \in D \setminus \{\bar{x}\},$$

where $k \geq 0$ is a constant. Prove that $\lim_{x\to\bar{x}} f(x)$ exists.

3.2.6 Determine the one-sided limits $\lim_{x\to 3^+} [x]$ and $\lim_{x\to 3^-} [x]$, where $[x]$ denotes the greatest integer that is less than or equal to x.

3.2.7 Find each of the following limits if they exist:

(a) $\lim_{x\to 1^+} \dfrac{x+1}{x-1}$.
(b) $\lim_{x\to 0^+} \left|x^3 \sin(1/x)\right|$.
(c) $\lim_{x\to 1}(x-[x])$.

3.2.8 For $a \in \mathbb{R}$, let f be the function given by

$$f(x) = \begin{cases} x^2, & \text{if } x > 1; \\ ax-1, & \text{if } x \le 1. \end{cases}$$

Find the value of a such that $\lim_{x\to 1} f(x)$ exists.

3.2.9 Determine all values of $\bar{x}$ such that the limit $\lim_{x\to\bar{x}}(1+x-[x])$ exists.

3.2.10 Let $a, b \in \mathbb{R}$ and suppose $f : (a,b) \to \mathbb{R}$ is increasing. Prove the following.

(a) If f is bounded above, then $\lim_{x\to b^-} f(x)$ exists and is a real number.
(b) If f is not bounded above, then $\lim_{x\to b^-} f(x) = \infty$.

State and prove analogous results in case f is bounded below and in case that the domain of f is one of $(-\infty, b)$, (a, ∞), or $(-\infty, \infty)$.

3.3 CONTINUITY

Definition 3.3.1 Let D be a nonempty subset of $\mathbb{R}$ and let $f : D \to \mathbb{R}$ be a function. The function f is said to be *continuous* at $x_0 \in D$ if for any real number $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in D$ and $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| < \varepsilon.$$

If f is continuous at every point $x \in D$, we say that f is *continuous on D* (or just continuous if no confusion occurs).

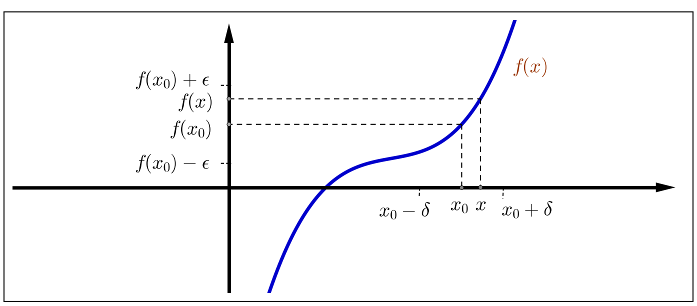

Figure 3.1: Definition of continuity.

■ **Example 3.3.1** Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3x+7$. Let $x_0 \in \mathbb{R}$ and let $\varepsilon > 0$. Choose $\delta = \varepsilon/3$. Then if $|x - x_0| < \delta$, we have

$$|f(x) - f(x_0)| = |3x+7-(3x_0+7)| = |3(x-x_0)| = 3|x-x_0| < 3\delta = \varepsilon.$$

This shows that f is continuous at x_0.

Remark 3.3.1 Note that the above definition of continuity does not mention limits. This allows to include in the definition, points $x_0 \in D$ which are not limit points of D. If x_0 is an isolated point of D, then there is $\delta > 0$ such that $B(x_0;\delta) \cap D = \{x_0\}$. It follows that for $x \in B(x_0;\delta) \cap D$, $|f(x) - f(x_0)| = 0 < \varepsilon$ for any epsilon. Therefore, every function is continuous at an isolated point of its domain.

To study continuity at limit points of D, we have the following theorem which follows directly from the definitions of continuity and limit.

Theorem 3.3.2 Let $f\colon D \to \mathbb{R}$ and let $x_0 \in D$ be a limit point of D. Then f is continuous at x_0 if and only if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

■ **Example 3.3.2** Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3x^2 - 2x + 1$. Fix $x_0 \in \mathbb{R}$. Since, from the results of the previous theorem, we have

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} (3x^2 - 2x + 1) = 3x_0^2 - 2x_0 + 1 = f(x_0),$$

it follows that f is continuous at x_0.

The following theorem follows directly from the definition of continuity, Theorem 3.1.2 and Theorem 3.3.2 and we leave its proof as an exercise.

Theorem 3.3.3 Let $f\colon D \to \mathbb{R}$ and let $x_0 \in D$. Then f is continuous at x_0 if and only if for any sequence $\{x_k\}$ in D that converges to x_0, the sequence $\{f(x_k)\}$ converges to $f(x_0)$.

The proofs of the next two theorems are straightforward using Theorem 3.3.3.

Theorem 3.3.4 Let $f, g\colon D \to \mathbb{R}$ and let $x_0 \in D$. Suppose f and g are continuous at x_0. Then

(a) $f + g$ and fg are continuous at x_0.

(b) cf is continuous at x_0 for any constant c.

(c) If $g(x_0) \neq 0$, then $\dfrac{f}{g}$ (defined on $\widetilde{D} = \{x \in D : g(x) \neq 0\}$) is continuous at x_0.

Proof: We prove (a) and leave the other parts as an exercise. We will use Theorem 3.3.3. Let $\{x_k\}$ be a sequence in D that converges to x_0. Since f and g are continuous at x_0, by Theorem 3.3.3 we obtain that $\{f(x_k)\}$ converges to $f(x_0)$ and $\{g(x_k)\}$ converges to $g(x_0)$. By Theorem 2.2.1 (a),we get that $\{f(x_k) + g(x_k)\}$ converges to $f(x_0) + g(x_0)$. Therefore,

$$\lim_{k \to \infty} (f+g)(x_k) = \lim_{k \to \infty} f(x_k) + g(x_k) = f(x_0) + g(x_0) = (f+g)(x_0).$$

Since $\{x_k\}$ was arbitrary, using Theorem 3.3.3 again we conclude $f + g$ is continuous at x_0. □

Theorem 3.3.5 Let $f\colon D \to \mathbb{R}$ and let $g\colon E \to \mathbb{R}$ with $f(D) \subset E$. If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Exercises

3.3.1 Prove, using definition 3.3.1, that each of the following functions is continuous on the given domain:

(a) $f(x) = ax + b$, $a, b \in \mathbb{R}$, on $\mathbb{R}$.
(b) $f(x) = x^2 - 3$ on $\mathbb{R}$.
(c) $f(x) = \sqrt{x}$ on $[0, \infty)$.
(d) $f(x) = \dfrac{1}{x}$ on $\mathbb{R} \setminus \{0\}$.

3.3.2 Determine the values of x at which each function is continuous. The domain of all the functions is $\mathbb{R}$.

(a) $f(x) = \begin{cases} \left|\dfrac{\sin x}{x}\right|, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$

(b) $f(x) = \begin{cases} \dfrac{\sin x}{|x|}, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$

(c) $f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$

(d) $f(x) = \begin{cases} \cos \dfrac{\pi x}{2}, & \text{if } |x| \leq 1; \\ |x - 1|, & \text{if } |x| > 1. \end{cases}$

(e) $f(x) = \lim_{n\to\infty} \sin \dfrac{\pi}{2(1 + x^{2n})}, \quad x \in \mathbb{R}.$

3.3.3 Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} x^2 + a, & \text{if } x > 2; \\ ax - 1, & \text{if } x \leq 2. \end{cases}$$

Find the value of a such that f is continuous.

3.3.4 Let $f : D \to \mathbb{R}$ and let $x_0 \in D$. Prove that if f is continuous at x_0, then $|f|$ is continuous at this point. Is the converse true in general?

3.3.5 Prove Theorem 3.3.3. (*Hint:* treat separately the cases when x_0 is a limit point of D and when it is not.)

3.3.6 Prove parts (b) and (c) of Theorem 3.3.4.

3.3.7 Prove Theorem 3.3.5.

3.3.8 ▶ Explore the continuity of the function f in each case below.

(a) Let $g,h:[0,1]\to\mathbb{R}$ be continuous functions and define

$$f(x)=\begin{cases} g(x), & \text{if } x\in\mathbb{Q}\cap[0,1];\\ h(x), & \text{if } x\in\mathbb{Q}^c\cap[0,1].\end{cases}$$

Prove that if $g(a)=h(a)$, for some $a\in[0,1]$, then f is continuous at a.

(b) Let $f\colon[0,1]\to\mathbb{R}$ be the function given by

$$f(x)=\begin{cases} x, & \text{if } x\in\mathbb{Q}\cap[0,1];\\ 1-x, & \text{if } x\in\mathbb{Q}^c\cap[0,1].\end{cases}$$

Find all the points on $[0,1]$ at which the function is continuous.

3.3.9 ▷ Consider the Thomae function defined on $(0,1]$ by

$$f(x)=\begin{cases} \frac{1}{q}, & \text{if } x=\frac{p}{q}, p,q\in\mathbb{N}, \text{where } p \text{ and } q \text{ have no common factors;}\\ 0, & \text{if } x \text{ is irrational.}\end{cases}$$

(a) Prove that for every $\varepsilon>0$, the set

$$A_\varepsilon=\{x\in(0,1]: f(x)\geq\varepsilon\}$$

is finite.

(b) Prove that f is continuous at every irrational point, and discontinuous at every rational point.

3.3.10 ▷ Consider k distinct points $x_1,x_2,\ldots,x_k\in\mathbb{R}$, $k\geq 1$. Find a function defined on $\mathbb{R}$ that is continuous at each x_i, $i=1,\ldots,k$, and discontinuous at all other points.

3.3.11 Suppose that f,g are continuous functions on $\mathbb{R}$ and $f(x)=g(x)$ for all $x\in\mathbb{Q}$. Prove that $f(x)=g(x)$ for all $x\in\mathbb{R}$.

3.4 PROPERTIES OF CONTINUOUS FUNCTIONS

Recall from Definition 2.6.3 that a subset A of $\mathbb{R}$ is compact if and only if every sequence $\{a_n\}$ in A has a subsequence $\{a_{n_k}\}$ that converges to a point $a\in A$.

Theorem 3.4.1 Let D be a nonempty compact subset of $\mathbb{R}$ and let $f\colon D\to\mathbb{R}$ be a continuous function. Then $f(D)$ is a compact subset of $\mathbb{R}$. In particular, $f(D)$ is closed and bounded.

Proof: Take any sequence $\{y_n\}$ in $f(D)$. Then for each n, there exists $a_n\in D$ such that $y_n=f(a_n)$. Since D is compact, there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ and a point $a\in D$ such that

$$\lim_{k\to\infty} a_{n_k}=a\in D.$$

It now follows from Theorem 3.3.3 that

$$\lim_{k\to\infty} y_{n_k}=\lim_{k\to\infty} f(a_{n_k})=f(a)\in f(D).$$

Therefore, $f(D)$ is compact.

The final conclusion follows from Theorem 2.6.5 □

Definition 3.4.1 We say that the function $f\colon D \to \mathbb{R}$ has an *absolute minimum* at $\bar{x} \in D$ if

$f(x) \geq f(\bar{x})$ for every $x \in D$.

Similarly, we say that f has an *absolute maximum* at $\bar{x}$ if

$f(x) \leq f(\bar{x})$ for every $x \in D$.

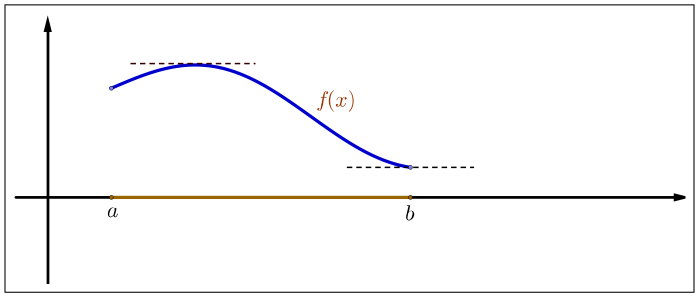

Figure 3.2: Absolute maximum and absolute minimum of f on $[a,b]$.

Theorem 3.4.2 — Extreme Value Theorem. Suppose $f\colon D \to \mathbb{R}$ is continuous and D is a compact set. Then f has an absolute minimum and an absolute maximum on D.

Proof: Since D is compact, $A = f(D)$ is closed and bounded (see Theorem 2.6.5). Let

$$m = \inf A = \inf_{x \in D} f(x).$$

In particular, $m \in \mathbb{R}$. For every $n \in \mathbb{N}$, there exists $a_n \in A$ such that

$m \leq a_n < m + 1/n$.

For each n, since $a_n \in A = f(D)$, there exists $x_n \in D$ such that $a_n = f(x_n)$ and, hence,

$$m \leq f(x_n) < m + 1/n.$$

By the compactness of D, there exists an element $\bar{x} \in D$ and a subsequence $\{x_{n_k}\}$ that converges to $\bar{x} \in D$ as $k \to \infty$. Because

$$m \leq f(x_{n_k}) < m + \frac{1}{n_k} \text{ for every } k,$$

by the squeeze theorem (Theorem 2.1.6) we conclude $\lim_{k\to\infty} f(x_{n_k}) = m$. On the other hand, by continuity we have $\lim_{k\to\infty} f(x_{n_k}) = f(\bar{x})$. We conclude that $f(\bar{x}) = m \leq f(x)$ for every $x \in D$. Thus, f has an absolute minimum at $\bar{x}$. The proof is similar for the case of absolute maximum. □

Remark 3.4.3 The proof of Theorem 3.4.2 can be shortened by applying Theorem 2.6.4. However, we have provided a direct proof instead.

Corollary 3.4.4 If $f\colon [a,b] \to \mathbb{R}$ is continuous, then it has an absolute minimum and an absolute maximum on $[a,b]$.

Corollary 3.4.4 is sometimes referred to as the Extreme Value Theorem. It follows immediately from Theorem 3.4.2, and the fact that the interval $[a,b]$ is compact (see Example 2.6.4).

The following result is a basic property of continuous functions that is used in a variety of situations.

Lemma 3.4.5 Let $f\colon D \to \mathbb{R}$ be continuous at $c \in D$. Suppose $f(c) > 0$. Then there exists $\delta > 0$ such that

$$f(x) > 0 \text{ for every } x \in B(c;\delta) \cap D.$$

Proof: Let $\varepsilon = f(c) > 0$. By the continuity of f at c, there exists $\delta > 0$ such that if $x \in D$ and $|x-c| < \delta$, then

$$|f(x) - f(c)| < \varepsilon.$$

This implies, in particular, that $f(x) > f(c) - \varepsilon = 0$ for every $x \in B(c;\delta) \cap D$. The proof is now complete. □

Remark 3.4.6 An analogous result holds if $f(c) < 0$.

Theorem 3.4.7 Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Suppose $f(a) \cdot f(b) < 0$ (this means either $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$). Then there exists $c \in (a,b)$ such that $f(c) = 0$.

Proof: We prove only the case $f(a) < 0 < f(b)$ (the case $f(a) > 0 > f(b)$ is completely analogous). Define

$$A = \{x \in [a,b] : f(x) \leq 0\}.$$

This set is nonempty since $a \in A$. This set is also bounded since $A \subset [a,b]$. Therefore, $c = \sup A$ exists and $a \leq c \leq b$. We are going to prove that $f(c) = 0$ by showing that $f(c) < 0$ and $f(c) > 0$ lead to contradictions.

Suppose $f(c) < 0$. Then there exists $\delta > 0$ such that

$$f(x) < 0 \text{ for all } x \in B(c;\delta) \cap [a,b].$$

Because $c < b$ (since $f(b) > 0$), we can find $s \in (c,b)$ such that $f(s) < 0$ (indeed $s = \min\{c + \delta/2, (c+b)/2\}$ will do). This is a contradiction because $s \in A$ and $s > c$.

Suppose $f(c) > 0$. Then there exists $\delta > 0$ such that

$$f(x) > 0 \text{ for all } x \in B(c;\delta) \cap [a,b].$$

Since $a < c$ (because $f(a) < 0$), there exists $t \in (a,c)$ such that $f(x) > 0$ for all $x \in (t,c)$ (in fact, $t = \max\{c - \delta/2, (a+c)/2\}$ will do). On the other hand, since $t < c = \sup A$, there exists $t' \in A$ with $t < t' \leq c$. But then $t < t'$ and $f(t') \leq 0$. This is a contradiction. We conclude that $f(c) = 0$. □

Theorem 3.4.8 — Intermediate Value Theorem. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Suppose $f(a) < \gamma < f(b)$. Then there exists a number $c \in (a,b)$ such that $f(c) = \gamma$.

The same conclusion follows if $f(a) > \gamma > f(b)$.

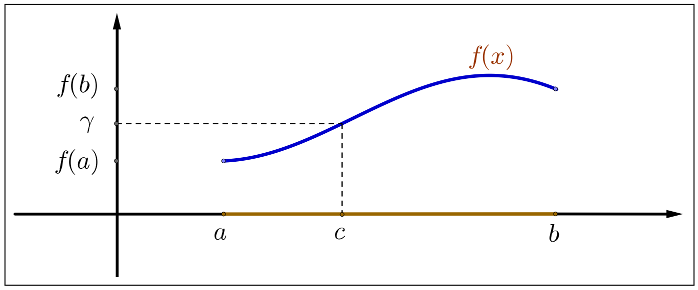

Figure 3.3: Illustration of the Intermediate Value Theorem.

Proof: Define

$$\varphi(x) = f(x) - \gamma,\ x \in [a,b].$$

Then φ is continuous on $[a,b]$. Moreover,

$$\varphi(a)\varphi(b) = [f(a) - \gamma][f(b) - \gamma] < 0.$$

By Theorem 3.4.7, there exists $c \in (a,b)$ such that $\varphi(c) = 0$. This is equivalent to $f(c) = \gamma$. The proof is now complete. $\square$

Corollary 3.4.9 Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Let

$$m = \min\{f(x) : x \in [a,b]\} \text{ and } M = \max\{f(x) : x \in [a,b]\}.$$

Then for every $\gamma \in [m,M]$, there exists $c \in [a,b]$ such that $f(c) = \gamma$.

■ **Example 3.4.1** We will use the Intermediate Value Theorem to prove that the equation $e^x = -x$ has at least one real solution. We will assume known that the exponential function is continuous on $\mathbb{R}$ and that $e^x < 1$ for $x < 0$.

First define the function $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = e^x + x$. Notice that the given equation has a solution x if and only if $f(x) = 0$. Now, the function f is continuous (as the sum of continuous functions). Moreover, note that $f(-1) = e^{-1} + (-1) < 1 - 1 = 0$ and $f(0) = 1 > 0$. We can now apply the Intermediate Value Theorem to the function f on the interval $[-1,0]$ with $\gamma = 0$ to conclude that there is $c \in [-1,0]$ such that $f(c) = 0$. The point c is the desired solution to the original equation.

■ **Example 3.4.2** We show now that, given $n \in \mathbb{N}$, every positive real number has a positive n-th root. Let $n \in \mathbb{N}$ and let $a \in \mathbb{R}$ with $a > 0$. First observe that $(1+a)^n \geq 1 + na > a$ (see Exercise 1.3.7). Now consider the function $f\colon [0,\infty) \to \mathbb{R}$ given by $f(x) = x^n$. Since $f(0) = 0$ and $f(1+a) > a$, it follows from the Intermediate Value Theorem that there is $x \in (0, 1+a)$ such that $f(x) = a$. That is, $x^n = a$, as desired. (We show later in Example 4.3.1 that such an x is unique.)

We present below a second proof of Theorem 3.4.8 that does not depend on Theorem 3.4.7, but, instead, relies on the Nested Intervals Theorem (Theorem 2.3.3).

Second Proof of Theorem 3.4.8: We construct a sequence of nested intervals as follows. Set $a_1 = a$, $b_1 = b$, and let $I_1 = [a,b]$. Let $c_1 = (a+b)/2$. If $f(c_1) = \gamma$, we are done. Otherwise, either

$$f(c_1) > \gamma \qquad \text{or}$$
$$f(c_1) < \gamma.$$

In the first case, set $a_2 = a_1$ and $b_1 = c_1$. In the second case, set $a_2 = c_1$ and $b_2 = b_1$. Now set $I_2 = [a_2, b_2]$. Note that in either case,

$$f(a_2) < \gamma < f(b_2).$$

Set $c_2 = (a_2 + b_2)/2$. If $f(c_2) = \gamma$, again we are done. Otherwise, either

$$f(c_2) > \gamma \qquad \text{or}$$
$$f(c_2) < \gamma.$$

In the first case, set $a_3 = a_2$ and $b_3 = c_2$. In the second case, set $a_3 = c_2$ and $b_3 = b_2$. Now set $I_3 = [a_3, b_3]$. Note that in either case,

$$f(a_3) < \gamma < f(b_3).$$

Proceeding in this way, either we find some c_{n_0} such that $f(c_{n_0}) = \gamma$ and, hence, the proof is complete, or we construct a sequence of closed bounded intervals $\{I_n\}$ with $I_n = [a_n, b_n]$ such that for all n,
(i) $I_n \supset I_{n+1}$,
(ii) $b_n - a_n = (b-a)/2^{n-1}$, and
(iii) $f(a_n) < \gamma < f(b_n)$.
In this case, we proceed as follows. Condition (ii) implies that $\lim_{n\to\infty}(b_n - a_n) = 0$. By the Nested Intervals Theorem (Theorem 2.3.3, part (b)), there exists $c \in [a,b]$ such that $\bigcap_{n=1}^{\infty} I_n = \{c\}$. Moreover, as we see from the proof of that theorem, $a_n \to c$ and $b_n \to c$ as $n \to \infty$.

By the continuity of f, we get

$$\lim_{n\to\infty} f(a_n) = f(c) \qquad \text{and}$$
$$\lim_{n\to\infty} f(b_n) = f(c).$$

Since $f(a_n) < \gamma < f(b_n)$ for all n, condition (iii) above and Theorem 2.1.5 give

$$f(c) \le \gamma \qquad \text{and}$$
$$f(c) \ge \gamma.$$

It follows that $f(c) = \gamma$. Note that, since $f(a) < \gamma < f(b)$, then $c \in (a,b)$. The proof is now complete. $\square$

Now we are going to discuss the continuity of the inverse function. For a function $f: D \to E$, where E is a subset of $\mathbb{R}$, we can define the new function $f: D \to \mathbb{R}$ by the same function notation. The function $f: D \to E$ is said to be continuous at a point $\bar{x} \in D$ if the corresponding function $f: D \to \mathbb{R}$ is continuous at $\bar{x}$.

Theorem 3.4.10 Let $f: [a,b] \to \mathbb{R}$ be strictly increasing and continuous on $[a,b]$. Let $c = f(a)$ and $d = f(b)$. Then f is one-to-one, $f([a,b]) = [c,d]$, and the inverse function f^{-1} defined on $[c,d]$ by

$$f^{-1}(f(x)) = x \text{ where } x \in [a,b],$$

is a continuous function from $[c,d]$ onto $[a,b]$.

Proof: The first two assertions follow from the monotonicity of f and the Intermediate Value Theorem (see also Corollary 3.4.9). We will prove that f^{-1} is continuous on $[c,d]$. Fix any $\bar{y} \in [c,d]$ and fix any sequence $\{y_k\}$ in $[c,d]$ that converges to $\bar{y}$. Let $\bar{x} \in [a,b]$ and $x_k \in [a,b]$ be such that

$$f(\bar{x}) = \bar{y} \text{ and } f(x_k) = y_k \text{ for every } k.$$

Then $f^{-1}(\bar{y}) = \bar{x}$ and $f^{-1}(y_k) = x_k$ for every k. Suppose by contradiction that $\{x_k\}$ does not converge to $\bar{x}$. Then there exist $\varepsilon_0 > 0$ and a subsequence $\{x_{k_\ell}\}$ of $\{x_k\}$ such that

$$|x_{k_\ell} - \bar{x}| \geq \varepsilon_0 \text{ for every } \ell. \tag{3.7}$$

Since the sequence $\{x_{k_\ell}\}$ is bounded, it has a further subsequence that converges to $x_0 \in [a,b]$. To simplify the notation, we will again call the new subsequence $\{x_{k_\ell}\}$. Taking limits in (3.7), we get

$$|x_0 - \bar{x}| \geq \varepsilon_0 > 0. \tag{3.8}$$

On the other hand, by the continuity of f, $\{f(x_{k_\ell})\}$ converges to $f(x_0)$. Since $f(x_{k_\ell}) = y_{k_\ell} \to \bar{y}$ as $\ell \to \infty$, it follows that $f(x_0) = \bar{y} = f(\bar{x})$. This implies $x_0 = \bar{x}$, which contradicts (3.8). $\square$

Remark 3.4.11 A similar result holds if the domain of f is the open interval (a,b) with some additional considerations. If $f\colon (a,b) \to \mathbb{R}$ is increasing and bounded, following the argument in Theorem 3.2.4 we can show that both $\lim_{x\to a^+} f(x) = c$ and $\lim_{x\to b^-} f(x) = d$ exist in $\mathbb{R}$ (see Exercise 3.2.10). Using the Intermediate Value Theorem we obtain that $f((a,b)) = (c,d)$. We can now proceed as in the previous theorem to show that f has a continuous inverse from (c,d) to (a,b).

If : $(a,b) \to \mathbb{R}$ is increasing, continuous, bounded below, but not bounded above, then $\lim_{x\to a^+} f(x) = c \in \mathbb{R}$, but $\lim_{x\to b^-} f(x) = \infty$ (again see Exercise 3.2.10). In this case we can show using the Intermediate Value Theorem that $f((a,b)) = (c,\infty)$ and we can proceed as above to prove that f has a continuous inverse from (c,∞) to (a,b).

The other possibilities lead to similar results.

A similar theorem can be proved for strictly decreasing functions.

Exercises

3.4.1 Let $f\colon D \to \mathbb{R}$ be continuous at $c \in D$ and let $\gamma \in \mathbb{R}$. Suppose $f(c) > \gamma$. Prove that there exists $\delta > 0$ such that

$$f(x) > \gamma \text{ for every } x \in B(c;\delta) \cap D.$$

3.4.2 Let f, g be continuous functions on $[a,b]$. Suppose $f(a) < g(a)$ and $f(b) > g(b)$. Prove that there exists $x_0 \in (a,b)$ such that $f(x_0) = g(x_0)$.

3.4.3 Prove that the equation $\cos x = x$ has at least on solution in $\mathbb{R}$. (Assume known that the function $\cos x$ is continuous.)

3.4.4 Prove that the equation $x^2 - 2 = \cos(x+1)$ has at least two real solutions. (Assume known that the function $\cos x$ is continuous.)

3.4.5 Let $f\colon [a,b] \to [a,b]$ be a continuous function.

(a) Prove that the equation $f(x) = x$ has a solution on $[a,b]$.

(b) Suppose further that

$$|f(x) - f(y)| < |x-y| \text{ for all } x,y \in [a,b], x \neq y.$$

Prove that the equation $f(x) = x$ has a unique solution on $[a,b]$.

3.4.6 ▷ Let f be a continuous function on $[a,b]$ and $x_1, x_2, \ldots, x_n \in [a,b]$. Prove that there exists $c \in [a,b]$ with

$$f(c) = \frac{f(x_1) + f(x_2) + \cdots f(x_n)}{n}.$$

3.4.7 ▷ Suppose f is a continuous function on $\mathbb{R}$ such that

$$|f(x)| < |x| \text{ for all } x \neq 0.$$

(a) Prove that $f(0) = 0$.

(b) Given two positive numbers a and b with $a < b$, prove that there exists $\ell \in [0,1)$ such that

$$|f(x)| \leq \ell |x| \text{ for all } x \in [a,b].$$

3.4.8 ▶ Let $f, g\colon [0,1] \to [0,1]$ be continuous functions such that

$$f(g(x)) = g(f(x)) \text{ for all } x \in [0,1].$$

Suppose further that f is monotone. Prove that there exists $x_0 \in [0,1]$ such that

$$f(x_0) = g(x_0) = x_0.$$

3.5 UNIFORM CONTINUITY

We discuss here a stronger notion of continuity.

Definition 3.5.1 Let D be a nonempty subset of $\mathbb{R}$. A function $f\colon D \to \mathbb{R}$ is called *uniformly continuous* on D if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $u, v \in D$ and $|u - v| < \delta$, then

$$|f(u) - f(v)| < \varepsilon.$$

■ **Example 3.5.1** Any constant function $f\colon D \to \mathbb{R}$, is uniformly continuous on its domain. Indeed, given $\varepsilon > 0$, $|f(u) - f(v)| = 0 < \varepsilon$ for all $u, v \in D$ regardless of the choice of δ.

The following result is straightforward from the definition.

Theorem 3.5.1 If $f\colon D \to \mathbb{R}$ is uniformly continuous on D, then f is continuous at every point $x_0 \in D$.

■ **Example 3.5.2** Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 7x - 2$. We will show that f is uniformly continuous on $\mathbb{R}$.

Let $\varepsilon > 0$ and choose $\delta = \varepsilon/7$. Then, if $u, v \in \mathbb{R}$ and $|u - v| < \delta$, we have

$$|f(u) - f(v)| = |7u - 2 - (7v - 2)| = |7(u - v)| = 7|u - v| < 7\delta = \varepsilon.$$

■ **Example 3.5.3** Let $f\colon [-3,2] \to \mathbb{R}$ be given by $f(x) = x^2$. This function is uniformly continuous on $[-3,2]$.

Let $\varepsilon > 0$. First observe that for $u,v \in [-3,2]$ we have $|u+v| \le |u|+|v| \le 6$. Now set $\delta = \varepsilon/6$. Then, for $u,v \in [-3,2]$ satisfying $|u-v| < \delta$, we have

$$|f(u)-f(v)| = |u^2-v^2| = |u-v||u+v| \le 6|u-v| < 6\delta = \varepsilon.$$

■ **Example 3.5.4** Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \frac{x^2}{x^2+1}$. We will show that f is uniformly continuous on $\mathbb{R}$.

Let $\varepsilon > 0$. We observe first that

$$\left|\frac{u^2}{u^2+1} - \frac{v^2}{v^2+1}\right| = \left|\frac{u^2(v^2+1)-v^2(u^2+1)}{(u^2+1)(v^2+1)}\right| = \frac{|u-v||u+v|}{(u^2+1)(v^2+1)} \le \frac{|u-v|(|u|+|v|)}{(u^2+1)(v^2+1)}$$

$$\le \frac{|u-v|((u^2+1)+(v^2+1))}{(u^2+1)(v^2+1)} \le |u-v|\left(\frac{1}{v^2+1}+\frac{1}{u^2+1}\right) \le 2|u-v|,$$

(where we used that $|x| \le x^2+1$ for all $x \in \mathbb{R}$, which can be easily seen by considering separately the cases $|x| < 1$ and $|x| \ge 1$).

Now, set $\delta = \varepsilon/2$. In view of the previous calculation, given $u,v \in \mathbb{R}$ satisfying $|u-v| < \delta$ we have

$$|f(u)-f(v)| = \left|\frac{u^2}{u^2+1} - \frac{v^2}{v^2+1}\right| \le 2|u-v| < 2\delta = \varepsilon.$$

Definition 3.5.2 (Hölder continuity). Let D be a nonempty subset of $\mathbb{R}$. A function $f\colon D \to \mathbb{R}$ is said to be *Hölder continuous* if there are constants $\ell \ge 0$ and $\alpha > 0$ such that

$$|f(u)-f(v)| \le \ell|u-v|^\alpha \text{ for every } u,v \in D.$$

The number α is called the *Hölder exponent* of the function. If $\alpha = 1$, then the function f is called *Lipschitz continuous*.

Theorem 3.5.2 If a function $f\colon D \to \mathbb{R}$ is Hölder continuous, then it is uniformly continuous.

Proof: Since f is Hölder continuous, there are constants $\ell \ge 0$ and $\alpha > 0$ such that

$$|f(u)-f(v)| \le \ell|u-v|^\alpha \text{ for every } u,v \in D.$$

If $\ell = 0$, then f is constant and, thus, uniformly continuous. Suppose next that $\ell > 0$. For any $\varepsilon > 0$, let $\delta = \left(\frac{\varepsilon}{\ell}\right)^{1/\alpha}$. Then, whenever $u,v \in D$, with $|u-v| < \delta$ we have

$$|f(u)-f(v)| \le \ell|u-v|^\alpha < \ell\delta^\alpha = \varepsilon.$$

The proof is now complete. □

■ **Example 3.5.5** (1) Let $D = [a,\infty)$, where $a > 0$. Then the function $f(x) = \sqrt{x}$ is Lipschitz continuous on D and, hence, uniformly continuous on this set. Indeed, for any $u,v \in D$, we have

$$|f(u)-f(v)| = |\sqrt{u}-\sqrt{v}| = \frac{|u-v|}{\sqrt{u}+\sqrt{v}} \le \frac{1}{2\sqrt{a}}|u-v|,$$

which shows f is Lipschitz with $\ell = 1/(2\sqrt{a})$.

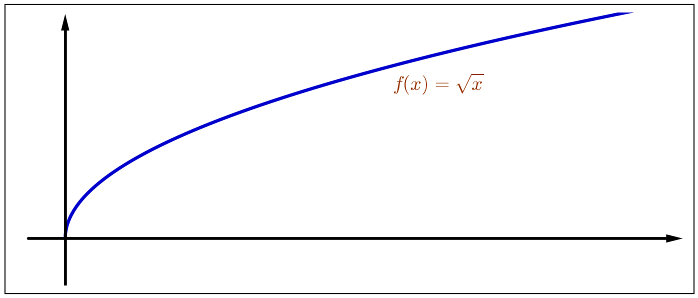

Figure 3.4: The square root function.

(2) Let $D = [0, \infty)$. Then f is not Lipschitz continuous on D, but it is Hölder continuous on D and, hence, f is also uniformly continuous on this set.

Indeed, suppose by contradiction that f is Lipschitz continuous on D. Then there exists a constant $\ell > 0$ such that

$$|f(u) - f(v)| = |\sqrt{u} - \sqrt{v}| \leq \ell |u - v| \text{ for every } u, v \in D.$$

Thus, for every $n \in \mathbb{N}$, we have

$$\left| \frac{1}{\sqrt{n}} - 0 \right| \leq \ell \left| \frac{1}{n} - 0 \right|.$$

This implies

$$\sqrt{n} \leq \ell \text{ or } n \leq \ell^2 \text{ for every } n \in \mathbb{N}.$$

This is a contradiction. Therefore, f is not Lipschitz continuous on D.

Let us show that f is Hölder continuous on D. We are going to prove that

$$|f(u) - f(v)| \leq |u - v|^{1/2} \text{ for every } u, v \in D. \tag{3.9}$$

The inequality in (3.9) holds obviously for $u = v = 0$. For $u > 0$ or $v > 0$, we have

$$\begin{aligned} |f(u) - f(v)| &= |\sqrt{u} - \sqrt{v}| \\ &= \left| \frac{u - v}{\sqrt{u} + \sqrt{v}} \right| \\ &= \sqrt{|u - v|} \, \frac{\sqrt{|u - v|}}{\sqrt{u} + \sqrt{v}} \\ &\leq \frac{\sqrt{|u| + |v|}}{\sqrt{u} + \sqrt{v}} \sqrt{|u - v|} \\ &= \sqrt{|u - v|}. \end{aligned}$$

Note that one can justify the inequality

$$\frac{\sqrt{|u| + |v|}}{\sqrt{u} + \sqrt{v}} \leq 1$$

by squaring both sides since they are both positive. Thus, (3.9) is satisfied.

While every uniformly continuous function on a set D is also continuous at each point of D, the converse is not true in general. The following example illustrates this point.

Example 3.5.6 Let $f\colon (0,1) \to \mathbb{R}$ be given by

$$f(x) = \frac{1}{x}.$$

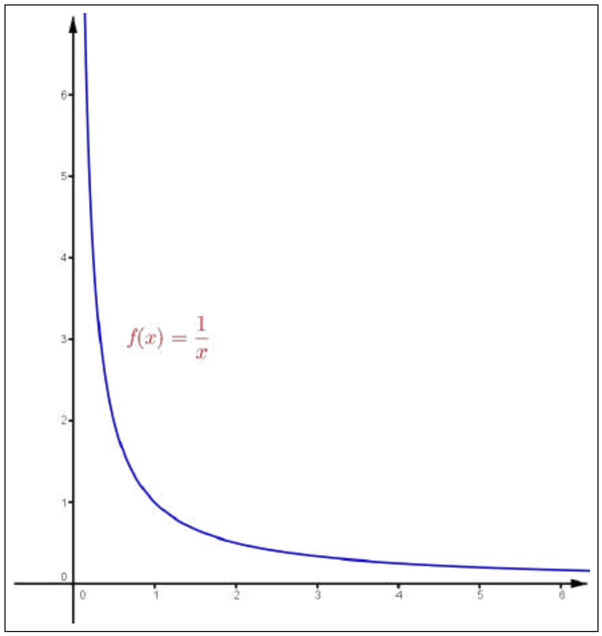

Figure 3.5: Continuous but not uniformly continuous on $(0,\infty)$.

We already know that this function is continuous at every $\bar{x} \in (0,1)$. We will show that f is not uniformly continuous on $(0,1)$. Let $\varepsilon = 2$ and $\delta > 0$. Set $\delta_0 = \min\{\delta/2, 1/4\}$, $x = \delta_0$, and $y = 2\delta_0$. Then $x, y \in (0,1)$ and $|x-y| = \delta_0 < \delta$, but

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y-x}{xy}\right| = \left|\frac{\delta_0}{2\delta_0^2}\right| = \left|\frac{1}{2\delta_0}\right| \geq 2 = \varepsilon.$$

This shows f is not uniformly continuous on $(0,1)$.

The following theorem offers a sequential characterization of uniform continuity analogous to that in Theorem 3.3.3.

Theorem 3.5.3 Let D be a nonempty subset of $\mathbb{R}$ and $f\colon D \to \mathbb{R}$. Then f is uniformly continuous on D if and only if the following condition holds

(C) for every two sequences $\{u_n\}$, $\{v_n\}$ in D such that $\lim_{n\to\infty}(u_n - v_n) = 0$, it follows that $\lim_{n\to\infty}(f(u_n) - f(v_n)) = 0$.

Proof: Suppose first that f is uniformly continuous and let $\{u_n\}$, $\{v_n\}$ be sequences in D such that $\lim_{n\to\infty}(u_n - v_n) = 0$. Let $\varepsilon > 0$. Choose $\delta > 0$ such that $|f(u) - f(v)| < \varepsilon$ whenever

$u, v \in D$ and $|u - v| < \delta$. Let $N \in \mathbb{N}$ be such that $|u_n - v_n| < \delta$ for $n \geq N$. For such n, we have $|f(u_n) - f(v_n)| < \varepsilon$. This shows $\lim_{n\to\infty}(f(u_n) - f(v_n)) = 0$.

To prove the converse, assume condition (C) holds and suppose, by way of contradiction, that f is not uniformly continuous. Then there exists $\varepsilon_0 > 0$ such that for any $\delta > 0$, there exist $u, v \in D$ with

$$|u - v| < \delta \text{ and } |f(u) - f(v)| \geq \varepsilon_0.$$

Thus, for every $n \in \mathbb{N}$, there exist $u_n, v_n \in D$ with

$$|u_n - v_n| \leq 1/n \text{ and } |f(u_n) - f(v_n)| \geq \varepsilon_0.$$

It follows that for such sequences, $\lim_{n\to\infty}(u_n - v_n) = 0$, but $\{f(u_n) - f(v_n)\}$ does not converge to zero, which contradicts the assumption. □

■ **Example 3.5.7** Using this theorem, we can give an easier proof that the function in Example 3.5.6 is not uniformly continuous. Consider the two sequences $u_n = 1/(n+1)$ and $v_n = 1/n$ for all $n \geq 2$. Then clearly, $\lim_{n\to\infty}(u_n - v_n) = 0$, but

$$\lim_{n\to\infty}(f(u_n) - f(v_n)) = \lim_{n\to\infty}\left(\frac{1}{1/(n+1)} - \frac{1}{1/n}\right) = \lim_{n\to\infty}(n+1-n) = 1 \neq 0.$$

The following theorem shows one important case in which continuity implies uniform continuity.

Theorem 3.5.4 Let $f\colon D \to \mathbb{R}$ be a continuous function. Suppose D is compact. Then f is uniformly continuous on D.

Proof: Suppose by contradiction that f is not uniformly continuous on D. Then there exists $\varepsilon_0 > 0$ such that for any $\delta > 0$, there exist $u, v \in D$ with

$$|u - v| < \delta \text{ and } |f(u) - f(v)| \geq \varepsilon_0.$$

Thus, for every $n \in \mathbb{N}$, there exist $u_n, v_n \in D$ with

$$|u_n - v_n| \leq 1/n \text{ and } |f(u_n) - f(v_n)| \geq \varepsilon_0.$$

Since D is compact, there exist $u_0 \in D$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$u_{n_k} \to u_0 \text{ as } k \to \infty.$$

Then

$$|u_{n_k} - v_{n_k}| \leq \frac{1}{n_k},$$

for all k and, hence, we also have

$$v_{n_k} \to u_0 \text{ as } k \to \infty.$$

By the continuity of f,

$$f(u_{n_k}) \to f(u_0) \text{ and } f(v_{n_k}) \to f(u_0).$$

Therefore, $\{f(u_{n_k}) - f(v_{n_k})\}$ converges to zero, which is a contradiction. The proof is now complete. □

We now prove a result that characterizes uniform continuity on open bounded intervals. We first make the observation that if $f: D \to \mathbb{R}$ is uniformly continuous on D and $A \subset D$, then f is uniformly continuous on A. More precisely, the restriction $f_{|A}: A \to \mathbb{R}$ is uniformly continuous on A (see Section 1.2 for the notation). This follows by noting that if $|f(u) - f(v)| < \varepsilon$ whenever $u, v \in D$ with $|u - v| < \delta$, then we also have $|f(u) - f(v)| < \varepsilon$ when we restrict u, v to be in A.

Theorem 3.5.5 Let $a, b \in \mathbb{R}$ and $a < b$. A function $f: (a,b) \to \mathbb{R}$ is uniformly continuous if and only if f can be extended to a continuous function $\tilde{f}: [a,b] \to \mathbb{R}$ (that is, there is a continuous function $\tilde{f}: [a,b] \to \mathbb{R}$ such that $f = \tilde{f}_{|(a,b)}$).

Proof: Suppose first that there exists a continuous function $\tilde{f}: [a,b] \to \mathbb{R}$ such that $f = \tilde{f}_{|(a,b)}$. By Theorem 3.5.4, the function $\tilde{f}$ is uniformly continuous on $[a,b]$. Therefore, it follows from our early observation that f is uniformly continuous on (a,b).

For the converse, suppose $f: (a,b) \to \mathbb{R}$ is uniformly continuous. We will show first that $\lim_{x\to a^+} f(x)$ exists. Note that the one sided limit corresponds to the limit in Theorem 3.2.2. We will check that the ε-δ condition of Theorem 3.2.2 holds.

Let $\varepsilon > 0$. Choose $\delta_0 > 0$ so that $|f(u) - f(v)| < \varepsilon$ whenever $u, v \in (a,b)$ and $|u - v| < \delta_0$. Set $\delta = \delta_0/2$. Then, if $u, v \in (a,b)$, $|u - a| < \delta$, and $|v - a| < \delta$ we have

$$|u - v| \le |u - a| + |a - v| < \delta + \delta = \delta_0$$

and, hence, $|f(u) - f(v)| < \varepsilon$. We can now invoke Theorem 3.2.2 to conclude $\lim_{x\to a^+} f(x)$ exists. In a similar way we can show that $\lim_{x\to b^-} f(x)$ exists. Now define, $\tilde{f}: [a,b] \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in (a,b); \\ \lim_{x\to a^+} f(x), & \text{if } x = a; \\ \lim_{x\to b^-} f(x), & \text{if } x = b. \end{cases}$$

By its definition $\tilde{f}_{|(a,b)} = f$ and, so, $\tilde{f}$ is continuous at every $x \in (a,b)$. Moreover, $\lim_{x\to a^+} \tilde{f}(x) = \lim_{x\to a^+} f(x) = \tilde{f}(a)$ and $\lim_{x\to b^-} \tilde{f}(x) = \lim_{x\to b^-} f(x) = \tilde{f}(b)$, so $\tilde{f}$ is also continuous at a and b by Theorem 3.3.2. Thus $\tilde{f}$ is the desired continuous extension of f. $\square$

Exercises

3.5.1 Prove that each of the following functions is uniformly continuous on the given domain:

(a) $f(x) = ax + b$, $a, b \in \mathbb{R}$, on $\mathbb{R}$.

(b) $f(x) = 1/x$ on $[a, \infty)$, where $a > 0$.

3.5.2 ▶ Prove that each of the following functions is not uniformly continuous on the given domain:

(a) $f(x) = x^2$ on $\mathbb{R}$.

(b) $f(x) = \sin \dfrac{1}{x}$ on $(0,1)$.

(c) $f(x) = \ln(x)$ on $(0, \infty)$.

3.5.3 Determine which of the following functions are uniformly continuous on the given domains.

(a) $f(x) = x\sin(\frac{1}{x})$ on $(0,1)$.

(b) $f(x) = \dfrac{x}{x+1}$ on $[0,\infty)$.

(c) $f(x) = \dfrac{1}{|x-1|}$ on $(0,1)$.

(d) $f(x) = \dfrac{1}{|x-2|}$ on $(0,1)$.

3.5.4 Let $D \subset \mathbb{R}$ and $k \in \mathbb{R}$. Prove that if $f,g\colon D \to \mathbb{R}$ are uniformly continuous on D, then $f+g$ and kf are uniformly continuous on D.

3.5.5 Give an example of a subset D of $\mathbb{R}$ and uniformly continuous functions $f,g\colon D \to \mathbb{R}$ such that fg is not uniformly conitnuous on D.

3.5.6 Let D be a nonempty subset of $\mathbb{R}$ and let $f\colon D \to \mathbb{R}$. Suppose that f is uniformly continuous on D. Prove that if $\{x_n\}$ is a Cauchy sequence with $x_n \in D$ for every $n \in \mathbb{N}$, then $\{f(x_n)\}$ is also a Cauchy sequence.

3.5.7 ▷ Let $a,b \in \mathbb{R}$ and let $f\colon (a,b) \to \mathbb{R}$.

(a) Prove that if f is uniformly continuous, then f is bounded.

(b) Prove that if f is continuous, bounded, and monotone, then it is uniformly continuous.

3.5.8 ▷ Let f be a continuous function on $[a,\infty)$. Suppose

$$\lim_{x\to\infty} f(x) = c.$$

(a) Prove that f is bounded on $[a,\infty)$.

(b) Prove that f is uniformly continuous on $[a,\infty)$.

(c) Suppose further that $c > f(a)$. Prove that there exists $x_0 \in [a,\infty)$ such that

$$f(x_0) = \inf\{f(x) : x \in [a,\infty)\}.$$

3.6 LIMIT SUPERIOR AND LIMIT INFERIOR OF FUNCTIONS

We extend to functions the concepts of limit superior and limit inferior.

Definition 3.6.1 Let $f\colon E \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Recall that

$$B_0(\bar{x};\delta) = B_-(\bar{x};\delta) \cup B_+(\bar{x};\delta) = (\bar{x}-\delta,\bar{x}) \cup (\bar{x},\bar{x}+\delta).$$

The *limit superior of the function f at $\bar{x}$* is defined by

$$\limsup_{x\to\bar{x}} f(x) = \inf_{\delta>0} \sup_{x\in B_0(\bar{x};\delta)\cap D} f(x).$$

Similarly, the *limit inferior of the function f at $\bar{x}$* is defined by

$$\liminf_{x\to\bar{x}} f(x) = \sup_{\delta>0} \inf_{x\in B_0(\bar{x};\delta)\cap D} f(x).$$

Consider the extended real-valued function $g\colon (0,\infty) \to (-\infty,\infty]$ defined by

$$g(\delta) = \sup_{x\in B_0(\bar{x};\delta)\cap D} f(x). \tag{3.10}$$

It is clear that g is increasing and

$$\limsup_{x\to\bar{x}} f(x) = \inf_{\delta>0} g(\delta).$$

We say that the function f is *locally bounded above around* $\bar{x}$ if there exists $\delta > 0$ and $M > 0$ such that

$$f(x) \le M \text{ for all } x \in B(\bar{x};\delta)\cap D.$$

Clearly, if f is locally bounded above around $\bar{x}$, then $\limsup_{x\to\bar{x}} f(x)$ is a real number, while $\limsup_{x\to\bar{x}} f(x) = \infty$ in the other case. Similar discussion applies for the limit inferior.

Theorem 3.6.1 Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Then $\ell = \limsup_{x\to\bar{x}} f(x)$ if and only if the following two conditions hold:

(1) For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(x) < \ell + \varepsilon \text{ for all } x \in B_0(\bar{x};\delta)\cap D;$$

(2) For every $\varepsilon > 0$ and for every $\delta > 0$, there exists $x_\delta \in B_0(\bar{x};\delta)\cap D$ such that

$$\ell - \varepsilon < f(x_\delta).$$

Proof: Suppose $\ell = \limsup_{x\to\bar{x}} f(x)$. Then

$$\ell = \inf_{\delta>0} g(\delta),$$

where g is defined in (3.10). For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\ell \le g(\delta) = \sup_{x\in B_0(\bar{x};\delta)\cap D} f(x) < \ell + \varepsilon.$$

Thus,

$$f(x) < \ell + \varepsilon \text{ for all } x \in B_0(\bar{x};\delta)\cap D,$$

which proves condition (1). Next note that for any $\varepsilon > 0$ and $\delta > 0$, we have

$$\ell - \varepsilon < \ell \le g(\delta) = \sup_{x\in B_0(\bar{x};\delta)\cap D} f(x).$$

Thus, there exists $x_\delta \in B_0(\bar{x};\delta)\cap D$ with

$$\ell - \varepsilon < f(x_\delta).$$

This proves (2).

Let us now prove the converse. Suppose (1) and (2) are satisfied. Fix any $\varepsilon > 0$ and let $\delta > 0$ satisfy (1). Then

$$g(\delta) = \sup_{x \in B_0(\bar{x};\delta)\cap D} f(x) \le \ell + \varepsilon.$$

This implies

$$\limsup_{x\to\bar{x}} f(x) = \inf_{\delta>0} g(\delta) \le \ell + \varepsilon.$$

Since ε is arbitrary, we get

$$\limsup_{x\to\bar{x}} f(x) \le \ell.$$

Again, let $\varepsilon > 0$. Given $\delta > 0$, let x_δ be as in (2). Therefore,

$$\ell - \varepsilon < f(x_\delta) \le \sup_{x \in B_0(\bar{x};\delta)\cap D} f(x) = g(\delta).$$

This implies

$$\ell - \varepsilon \le \inf_{\delta>0} g(\delta) = \limsup_{x\to\bar{x}} f(x).$$

It follows that $\ell \le \limsup_{x\to\bar{x}} f(x)$. Therefore, $\ell = \limsup_{x\to\bar{x}} f(x)$. $\square$

Corollary 3.6.2 Suppose $\ell = \limsup_{x\to\bar{x}} f(x)$. Then there exists a sequence $\{x_k\}$ in D such that $\{x_k\}$ converges to $\bar{x}$, $x_k \neq \bar{x}$ for every k, and

$$\lim_{k\to\infty} f(x_k) = \ell.$$

Moreover, if $\{y_k\}$ is a sequence in D that converges to $\bar{x}$, $y_k \neq \bar{x}$ for every k, and $\lim_{k\to\infty} f(y_k) = \ell'$, then $\ell' \le \ell$.

Proof: For each $k \in \mathbb{N}$, take $\varepsilon_k = \dfrac{1}{k}$. By (1) of Theorem 3.6.1, there exists $\delta_k > 0$ such that

$$f(x) < \ell + \varepsilon_k \text{ for all } x \in B_0(\bar{x};\delta_k)\cap D. \tag{3.11}$$

Let $\delta'_k = \min\{\delta_k, \frac{1}{k}\}$. Then $\delta'_k \le \delta_k$ and $\lim_{k\to\infty} \delta'_k = 0$. From (2) of Theorem 3.6.1, there exists $x_k \in B_0(\bar{x};\delta'_k)\cap D$ such that

$$\ell - \varepsilon_k < f(x_k).$$

Moreover, $f(x_k) < \ell + \varepsilon_k$ by (3.11). Therefore, $\{x_k\}$ is a sequence that satisfies the conclusion of the corollary.

Now let $\{y_k\}$ be a sequence in D that converges to $\bar{x}$, $y_k \neq \bar{x}$ for every k, and $\lim_{k\to\infty} f(y_k) = \ell'$. For any $\varepsilon > 0$, let $\delta > 0$ be as in (1) of Theorem 3.6.1. Since $y_k \in B_0(\bar{x};\delta)\cap D$ when k is sufficiently large, we have

$$f(y_k) < \ell + \varepsilon$$

for such k. This implies $\ell' \le \ell + \varepsilon$. It follows that $\ell' \le \ell$. $\square$

Remark 3.6.3 Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Suppose $\limsup_{x\to\bar{x}} f(x)$ is a real number. Define

$$A = \{\ell \in \mathbb{R} : \exists \{x_k\} \subset D, x_k \neq \bar{x} \text{ for every } k, x_k \to \bar{x}, f(x_k) \to \ell\}.$$

Then the previous corollary shows that $A \neq \emptyset$ and $\limsup_{x\to\bar{x}} f(x) = \max A$.

Theorem 3.6.4 Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Then

$$\limsup_{x\to\bar{x}} f(x) = \infty$$

if and only if there exists a sequence $\{x_k\}$ in D such that $\{x_k\}$ converges to $\bar{x}$, $x_k \neq \bar{x}$ for every k, and $\lim_{k\to\infty} f(x_k) = \infty$.

Proof: Suppose $\limsup_{x\to\bar{x}} f(x) = \infty$. Then

$$\inf_{\delta>0} g(\delta) = \infty,$$

where g is the extended real-valued function defined in (3.10). Thus, $g(\delta) = \infty$ for every $\delta > 0$. Given $k \in \mathbb{N}$, for $\delta_k = \frac{1}{k}$, since

$$g(\delta_k) = \sup_{x\in B_0(\bar{x};\delta_k)\cap D} f(x) = \infty,$$

there exists $x_k \in B_0(\bar{x};\delta_k)\cap D$ such that $f(x_k) > k$. Therefore, $\lim_{k\to\infty} f(x_k) = \infty$.

Let us prove the converse. Since $\lim_{k\to\infty} f(x_k) = \infty$, for every $M \in \mathbb{R}$, there exists $K \in \mathbb{N}$ such that

$$f(x_k) \geq M \text{ for every } k \geq K.$$

For any $\delta > 0$, we have

$$x_k \in B_0(\bar{x};\delta)\cap D$$

whenever k is sufficiently large. Thus,

$$g(\delta) = \sup_{x\in B_0(\bar{x};\delta)\cap D} f(x) \geq M.$$

This implies $g(\delta) = \infty$, and hence $\limsup_{x\to\bar{x}} f(x) = \infty$. $\square$

Theorem 3.6.5 Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Then

$$\limsup_{x\to\bar{x}} f(x) = -\infty$$

if and only if for any sequence $\{x_k\}$ in D such that $\{x_k\}$ converges to $\bar{x}$, $x_k \neq \bar{x}$ for every k, it follows that $\lim_{k\to\infty} f(x_k) = -\infty$. The latter is equivalent to $\lim_{x\to\bar{x}} f(x) = -\infty$.

Following the same arguments, we can prove similar results for inferior limits of functions.

Theorem 3.6.6 Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Then $\ell = \liminf_{x\to\bar{x}} f(x)$ if and only if the following two conditions hold:

(1) For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\ell - \varepsilon < f(x) \text{ for all } x \in B_0(\bar{x};\delta) \cap D;$$

(2) For every $\varepsilon > 0$ and for every $\delta > 0$, there exists $x \in B_0(\bar{x};\delta) \cap D$ such that

$$f(x) < \ell + \varepsilon.$$

Corollary 3.6.7 Suppose $\ell = \liminf_{x\to\bar{x}} f(x)$. Then there exists a sequence $\{x_k\}$ in D such that x_k converges to $\bar{x}$, $x_k \neq \bar{x}$ for every k, and

$$\lim_{k\to\infty} f(x_k) = \ell.$$

Moreover, if $\{y_k\}$ is a sequence in D that converges to $\bar{x}$, $y_k \neq \bar{x}$ for every k, and $\lim_{k\to\infty} f(y_k) = \ell'$, then $\ell' \geq \ell$.

Remark 3.6.8 Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Suppose $\liminf_{x\to\bar{x}} f(x)$ is a real number. Define

$$B = \{\ell \in \mathbb{R} : \exists \{x_k\} \subset D, x_k \neq \bar{x} \text{ for every k}, x_k \to \bar{x}, f(x_k) \to \ell\}.$$

Then $B \neq \emptyset$ and $\liminf_{x\to\bar{x}} f(x) = \min B$.

Theorem 3.6.9 Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Then

$$\liminf_{x\to\bar{x}} f(x) = -\infty$$

if and only if there exists a sequence $\{x_k\}$ in D such that $\{x_k\}$ converges to $\bar{x}$, $x_k \neq \bar{x}$ for every k, and $\lim_{k\to\infty} f(x_k) = -\infty$.

Theorem 3.6.10 Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Then

$$\liminf_{x\to\bar{x}} f(x) = \infty$$

if and only if for any sequence $\{x_k\}$ in D such that $\{x_k\}$ converges to $\bar{x}$, $x_k \neq \bar{x}$ for every k, it follows that $\lim_{k\to\infty} f(x_k) = \infty$. The latter is equivalent to $\lim_{x\to\bar{x}} f(x) = \infty$.

Theorem 3.6.11 Let $f\colon D \to \mathbb{R}$ and let $\bar{x}$ be a limit point of D. Then

$$\lim_{x\to\bar{x}} f(x) = \ell$$

if and only if

$$\limsup_{x\to\bar{x}} f(x) = \liminf_{x\to\bar{x}} f(x) = \ell.$$

Proof: Suppose

$$\lim_{x\to\bar{x}} f(x) = \ell.$$

Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\ell - \varepsilon < f(x) < \ell + \varepsilon \text{ for all } x \in B_0(\bar{x};\delta) \cap D.$$

Since this also holds for every $0 < \delta' < \delta$, we get

$$\ell - \varepsilon < g(\delta') \le \ell + \varepsilon.$$

It follows that

$$\ell - \varepsilon \le \inf_{\delta'>0} g(\delta') \le \ell + \varepsilon.$$

Therefore, $\limsup_{x\to\bar{x}} f(x) = \ell$ since ε is arbitrary. The proof for the limit inferior is similar. The converse follows directly from (1) of Theorem 3.6.1 and Theorem 3.6.6. □

Exercises

3.6.1 Let $D \subset \mathbb{R}$, $f\colon D \to \mathbb{R}$, and $\bar{x}$ be a limit point of D. Prove that $\liminf_{x\to\bar{x}} f(x) \le \limsup_{x\to\bar{x}} f(x)$.

3.6.2 Find each of the following limits:

(a) $\limsup_{x\to 0} \sin\left(\dfrac{1}{x}\right)$.

(b) $\liminf_{x\to 0} \sin\left(\dfrac{1}{x}\right)$.

(c) $\limsup_{x\to 0} \dfrac{\cos x}{x}$.

(d) $\liminf_{x\to 0} \dfrac{\cos x}{x}$.

3.7 LOWER SEMICONTINUITY AND UPPER SEMICONTINUITY

The concept of semicontinuity is convenient for the study of maxima and minima of some discontinuous functions.

Definition 3.7.1 Let $f\colon D \to \mathbb{R}$ and let $\bar{x} \in D$. We say that f is *lower semicontinuous* (l.s.c.) at $\bar{x}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(\bar{x}) - \varepsilon < f(x) \text{ for all } x \in B(\bar{x};\delta) \cap D. \tag{3.12}$$

Similarly, we say that f is *upper semicontinuous* (u.s.c.) at $\bar{x}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(x) < f(\bar{x}) + \varepsilon \text{ for all } x \in B(\bar{x};\delta) \cap D.$$

It is clear that f is continuous at $\bar{x}$ if and only if f is lower semicontinuous and upper semicontinuous at this point.

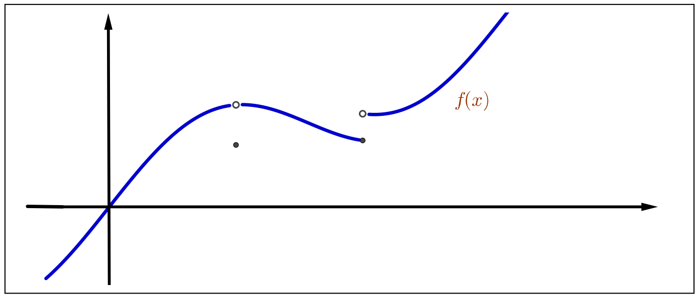

Figure 3.6: Lower semicontinuity.

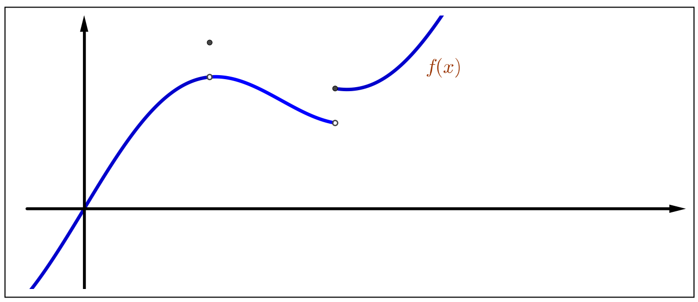

Figure 3.7: Upper semicontinuity.

Theorem 3.7.1 Let $f\colon D \to \mathbb{R}$ and let $\bar{x} \in D$ be a limit point of D. Then f is lower semicontinuous at $\bar{x}$ if and only if

$$\liminf_{x \to \bar{x}} f(x) \geq f(\bar{x}).$$

Similarly, f is upper semicontinuous at $\bar{x}$ if and only if

$$\limsup_{x \to \bar{x}} f(x) \leq f(\bar{x}).$$

Proof: Suppose f is lower semicontinuous at $\bar{x}$. Let $\varepsilon > 0$. Then there exists $\delta_0 > 0$ such that

$$f(\bar{x}) - \varepsilon < f(x) \text{ for all } x \in B(\bar{x}; \delta_0) \cap D.$$

This implies

$$f(\bar{x}) - \varepsilon \leq h(\delta_0),$$

where

$$h(\delta) = \inf_{x \in B_0(\bar{x}; \delta) \cap D} f(x).$$

Thus,

$$\liminf_{x \to \bar{x}} f(x) = \sup_{\delta > 0} h(\delta) \geq h(\delta_0) \geq f(\bar{x}) - \varepsilon.$$

Since ε is arbitrary, we obtain $\liminf_{x\to\bar{x}} f(x) \geq f(\bar{x})$.

We now prove the converse. Suppose

$$\liminf_{x\to\bar{x}} f(x) = \sup_{\delta>0} h(\delta) \geq f(\bar{x})$$

and let $\varepsilon > 0$. Since

$$\sup_{\delta>0} h(\delta) > f(\bar{x}) - \varepsilon,$$

there exists $\delta > 0$ such that $h(\delta) > f(\bar{x}) - \varepsilon$. This implies

$$f(x) > f(\bar{x}) - \varepsilon \text{ for all } x \in B_0(\bar{x};\delta) \cap D.$$

Since this is also true for $x = \bar{x}$, the function f is lower semicontinuous at $\bar{x}$.

The proof for the upper semicontinuous case is similar. □

Theorem 3.7.2 Let $f\colon D \to \mathbb{R}$ and let $\bar{x} \in D$. Then f is l.s.c. at $\bar{x}$ if and only if for every sequence $\{x_k\}$ in D that converges to $\bar{x}$,

$$\liminf_{k\to\infty} f(x_k) \geq f(\bar{x}).$$

Similarly, f is u.s.c. at $\bar{x}$ if and only if for every sequence $\{x_k\}$ in D that converges to $\bar{x}$,

$$\limsup_{k\to\infty} f(x_k) \leq f(\bar{x}).$$

Proof: Suppose f is l.s.c. at $\bar{x}$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that (3.12) holds. Since $\{x_k\}$ converges to $\bar{x}$, we have $x_k \in B(\bar{x};\delta)$ when k is sufficiently large. Thus,

$$f(\bar{x}) - \varepsilon < f(x_k)$$

for such k. It follows that $f(\bar{x}) - \varepsilon \leq \liminf_{k\to\infty} f(x_k)$. Since ε is arbitrary, it follows that $f(\bar{x}) \leq \liminf_{k\to\infty} f(x_k)$.

We now prove the converse. Suppose $\liminf_{k\to\infty} f(x_k) \geq f(\bar{x})$ and assume, by way of contradiction, that f is not l.s.c. at $\bar{x}$. Then there exists $\bar{\varepsilon} > 0$ such that for every $\delta > 0$, there exists $x_\delta \in B(\bar{x};\delta) \cap D$ with

$$f(\bar{x}) - \bar{\varepsilon} \geq f(x_\delta).$$

Applying this for $\delta_k = \dfrac{1}{k}$, we obtain a sequence $\{x_k\}$ in D that converges to $\bar{x}$ with

$$f(\bar{x}) - \bar{\varepsilon} \geq f(x_k) \text{ for every k.}$$

This implies

$$f(\bar{x}) - \bar{\varepsilon} \geq \liminf_{k\to\infty} f(x_k).$$

This is a contradiction. □

Definition 3.7.2 Let $f\colon D \to \mathbb{R}$. We say that f is *lower semicontinuous on* D (or lower semicontinuous if no confusion occurs) if it is lower semicontinuous at every point of D.

Theorem 3.7.3 Suppose D is a compact set of $\mathbb{R}$ and $f\colon D \to \mathbb{R}$ is lower semicontinuous. Then f has an absolute minimum on D. That means there exists $\bar{x} \in D$ such that

$$f(x) \geq f(\bar{x}) \text{ for all } x \in D.$$

Proof: We first prove that f is bounded below. Suppose by contradiction that for every $k \in \mathbb{N}$, there exists $x_k \in D$ such that

$$f(x_k) < -k.$$

Since D is compact, there exists a subsequence $\{x_{k_\ell}\}$ of $\{x_k\}$ that converges to $x_0 \in D$. Since f is l.s.c., by Theorem 3.7.2

$$\liminf_{\ell\to\infty} f(x_{k_\ell}) \geq f(x_0).$$

This is a contraction because $\liminf_{\ell\to\infty} f(x_{k_\ell}) = -\infty$. This shows f is bounded below. Define

$$\gamma = \inf\{f(x) : x \in D\}.$$

Since the set $\{f(x) : x \in D\}$ is nonempty and bounded below, $\gamma \in \mathbb{R}$.

Let $\{u_k\}$ be a sequence in D such that $\{f(u_k)\}$ converges to γ. By the compactness of D, the sequence $\{u_k\}$ has a convergent subsequence $\{u_{k_\ell}\}$ that converges to some $\bar{x} \in D$. Then

$$\gamma = \lim_{\ell\to\infty} f(u_{k_\ell}) = \liminf_{\ell\to\infty} f(u_{k_\ell}) \geq f(\bar{x}) \geq \gamma.$$

This implies $\gamma = f(\bar{x})$ and, hence,

$$f(x) \geq f(\bar{x}) \text{ for all } x \in D.$$

The proof is now complete. $\square$

The following theorem is proved similarly.

Theorem 3.7.4 Suppose D is a compact subset of $\mathbb{R}$ and $f\colon D \to \mathbb{R}$ is upper semicontinuous. Then f has an absolute maximum on D. That is, there exists $\bar{x} \in D$ such that

$$f(x) \leq f(\bar{x}) \text{ for all } x \in D.$$

For every $a \in \mathbb{R}$, define

$$\mathscr{L}_a(f) = \{x \in D : f(x) \leq a\}$$

and

$$\mathscr{U}_a(f) = \{x \in D : f(x) \geq a\}.$$

Theorem 3.7.5 Let $f\colon D \to \mathbb{R}$. Then f is lower semicontinuous if and only if $\mathscr{L}_a(f)$ is closed in D for every $a \in \mathbb{R}$. Similarly, f is upper semicontinuous if and only if $\mathscr{U}_a(f)$ is closed in D for every $a \in \mathbb{R}$.

Proof: Suppose f is lower semicontinuous. Using Corollary 2.6.10, we will prove that for every sequence $\{x_k\}$ in $\mathscr{L}_a(f)$ that converges to a point $\bar{x} \in D$, we get $\bar{x} \in \mathscr{L}_a(f)$. For every k, since $x_k \in \mathscr{L}_a(f)$, $f(x_k) \le a$.

Since f is lower semicontinuous at $\bar{x}$,

$$f(\bar{x}) \le \liminf_{k\to\infty} f(x_k) \le a.$$

Thus, $\bar{x} \in \mathscr{L}_a(f)$. It follows that $\mathscr{L}_a(f)$ is closed.

We now prove the converse. Fix any $\bar{x} \in D$ and $\varepsilon > 0$. Then the set

$$G = \{x \in D : f(x) > f(\bar{x}) - \varepsilon\} = D \setminus \mathscr{L}_{f(\bar{x})-\varepsilon(f)}$$

is open in D and $\bar{x} \in G$. Thus, there exists $\delta > 0$ such that

$$B(\bar{x};\delta) \cap D \subset G.$$

It follows that

$$f(\bar{x}) - \varepsilon < f(x) \text{ for all } x \in B(\bar{x};\delta) \cap D.$$

Therefore, f is lower semicontinuous. The proof for the upper semicontinuous case is similar. □

For every $a \in \mathbb{R}$, we also define

$$L_a(f) = \{x \in D : f(x) < a\}$$

and

$$U_a(f) = \{x \in D : f(x) > a\}.$$

Corollary 3.7.6 Let $f : D \to \mathbb{R}$. Then f is lower semicontinuous if and only if $U_a(f)$ is open in D for every $a \in \mathbb{R}$. Similarly, f is upper semicontinuous if and only if $L_a(f)$ is open in D for every $a \in \mathbb{R}$.

Theorem 3.7.7 Let $f : D \to \mathbb{R}$. Then f is continuous if and only if for every $a, b \in \mathbb{R}$ with $a < b$, the set

$$O_{a,b} = \{x \in D : a < f(x) < b\} = f^{-1}((a,b))$$

is an open set in D.

Proof: Suppose f is continuous. Then f is lower semicontinuous and upper semicontinuos. Fix $a, b \in \mathbb{R}$ with $a < b$. Then

$$O_{a,b} = L_b \cap U_a.$$

By Theorem 3.7.6, the set $O_{a,b}$ is open since it is the intersection of two open sets L_a and U_b.

Let us prove the converse. We will only show that f is lower semicontinuous since the proof of upper semicontinuity is similar. For every $a \in \mathbb{R}$, we have

$$U_a(f) = \{x \in D : f(x) > a\} = \cup_{n\in\mathbb{N}} f^{-1}((a, a+n))$$

Thus, $U_a(f)$ is open in D as it is a union of open sets in D. Therefore, f is lower semicontinuous by Corollary 3.7.6. □

Exercises

3.7.1 Let f be the function given by

$$f(x) = \begin{cases} x^2, & \text{if } x \neq 0; \\ -1, & \text{if } x = 0. \end{cases}$$

Prove that f is lower semicontinuous.

3.7.2 Let f be the function given by

$$f(x) = \begin{cases} x^2, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$$

Prove that f is upper semicontinuous.

3.7.3 Let $f, g \colon D \to \mathbb{R}$ be lower semicontinuous functions and let $k > 0$ be a constant. Prove that $f + g$ and kf are lower semicontinous functions on D.

3.7.4 ▶ Let $f \colon \mathbb{R} \to \mathbb{R}$ be a lower semicontinuous function such that

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty.$$

Prove that f has an absolute minimum at some $x_0 \in \mathbb{R}$.

4. DIFFERENTIATION

In this chapter, we discuss basic properties of the derivative of a function and several major theorems, including the Mean Value Theorem and l'Hôpital's Rule.

4.1 DEFINITION AND BASIC PROPERTIES OF THE DERIVATIVE

Let G be an open subset of $\mathbb{R}$ and consider a function $f\colon G \to \mathbb{R}$. For every $a \in G$, the function

$$\phi_a(x) = \frac{f(x) - f(a)}{x - a}$$

is defined on $G \setminus \{a\}$. Since G is an open set, a is a limit point of $G \setminus \{a\}$ (see Example 2.6.6). Thus, it is possible to discuss the limit

$$\lim_{x \to a} \phi_a(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Definition 4.1.1 Let G be an open subset of $\mathbb{R}$ and let $a \in G$. We say that the function f defined on G is *differentiable at a* if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists (as a real number). In this case, the limit is called the *derivative of f at a* denoted by $f'(a)$, and f is said to be *differentiable at a*. Thus, if f is differentiable at a, then

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

We say that f is *differentiable on G* if f is differentiable at every point $a \in G$. In this case, the function $f'\colon G \to \mathbb{R}$ is called the derivative of f on G.

■ **Example 4.1.1** (a) Let $f\colon \mathbb{R}\to\mathbb{R}$ be given by $f(x)=x$ and let $a\in\mathbb{R}$. Then

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{x-a}{x-a}=\lim_{x\to a}1=1.$$

It follows that f is differentiable at a and $f'(a)=1$.

(b) Let $f\colon \mathbb{R}\to\mathbb{R}$ be given by $f(x)=x^2$ and let $a\in\mathbb{R}$. Then

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{x^2-a^2}{x-a}=\lim_{x\to a}\frac{(x-a)(x+a)}{x-a}=\lim_{x\to a}(x+a)=2a.$$

Thus, f is differentiable at every $a\in\mathbb{R}$ and $f'(a)=2a$.

(c) Let $f\colon \mathbb{R}\to\mathbb{R}$ be given by $f(x)=|x|$ and let $a=0$. Then

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{|x|}{x}=\lim_{x\to 0^+}\frac{x}{x}=1,$$

and

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{|x|}{x}=\lim_{x\to 0^-}\frac{-x}{x}=-1.$$

Therefore, $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$ does not exist and, hence, f is not differentiable at 0.

Theorem 4.1.1 Let G be an open subset of $\mathbb{R}$ and let f be defined on G. If f is differentiable at $a\in G$, then f is continuous at this point.

Proof: We have the following identity for $x\in G\setminus\{a\}$:

$$\begin{aligned}f(x)&=f(x)-f(a)+f(a)\\&=\frac{f(x)-f(a)}{x-a}(x-a)+f(a).\end{aligned}$$

Thus,

$$\lim_{x\to a}f(x)=\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}(x-a)+f(a)\right]=f'(a)\cdot 0+f(a)=f(a).$$

Therefore, f is continuous at a by Theorem 3.3.2. □

Remark 4.1.2 The converse of Theorem 4.1.1 is not true. For instance, the absolute value function $f(x)=|x|$ is continuous at 0, but it is not differentiable at this point (as shown in the example above).

Theorem 4.1.3 Let G be an open subset of $\mathbb{R}$ and let $f,g\colon G\to\mathbb{R}$. Suppose both f and g are differentiable at $a\in G$. Then the following hold.

(a) The function $f+g$ is differentiable at a and

$$(f+g)'(a)=f'(a)+g'(a).$$

(b) For a constant c, the function cf is differentiable at a and

$$(cf)'(a)=cf'(a).$$

(c) The function fg is differentiable at a and

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

(d) Suppose additionally that $g(a) \neq 0$. Then the function $\frac{f}{g}$ is differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$

Proof: The proofs of (a) and (b) are straightforward and we leave them as exercises. Let us prove (c). For every $x \in G \setminus \{a\}$, we can write

$$\begin{aligned}\frac{(fg)(x) - (fg)(a)}{x-a} &= \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x-a} \\ &= \frac{(f(x) - f(a))g(x)}{x-a} + \frac{f(a)(g(x) - g(a))}{x-a}.\end{aligned}$$

By Theorem 4.1.1, the function g is continuous at a and, hence,

$$\lim_{x \to a} g(x) = g(a). \tag{4.1}$$

Thus,

$$\lim_{x \to a} \frac{(fg)(x) - (fg)(a)}{x-a} = f'(a)g(a) + f(a)g'(a).$$

This implies (c).

Next we show (d). Since $g(a) \neq 0$, by (4.1), there exists an open interval I containing a such that $g(x) \neq 0$ for all $x \in I$. Let $h = \frac{f}{g}$. Then h is defined on I. Moreover,

$$\begin{aligned}\frac{h(x) - h(a)}{x-a} &= \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(x)} + \frac{f(a)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} \\ &= \frac{\frac{1}{g(x)}(f(x) - f(a)) + \frac{f(a)}{g(x)g(a)}(g(a) - g(x))}{x-a} \\ &= \frac{1}{g(x)g(a)}\left[g(a)\frac{f(x) - f(a)}{x-a} - f(a)\frac{g(x) - g(a)}{x-a}\right].\end{aligned}$$

Taking the limit as $x \to a$, we obtain (d). The proof is now complete. □

■ **Example 4.1.2** Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$ and let $a \in \mathbb{R}$. Using Example 4.1.1(a) and Theorem 4.1.3(c) we can provide an alternative derivation of a formula for $f'(a)$. Indeed, let $g\colon \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x$. Then $f = g \cdot g$ so

$$f'(a) = (gg)'(a) = g'(a)g(a) + g(a)g'(a) = 2g'(a)g(a) = 2a.$$

Proceeding by induction, we can obtain the derivative of $g\colon \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^n$ for $n \in \mathbb{N}$ as $g'(a) = nx^{n-1}$. Furthermore, using this and Theorem 4.1.3(a)(b) we obtain the familiar formula for the derivative of a polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$ as $p'(x) = na_n x^{n-1} + \cdots + 2a_2 x + a_1$.

The following lemma is very convenient for studying the differentiability of the composition of functions.

Lemma 4.1.4 Let G be an open subset of $\mathbb{R}$ and let $f\colon G \to \mathbb{R}$. Suppose f is differentiable at a. Then there exists a function $u\colon G \to \mathbb{R}$ satisfying

$$f(x) - f(a) = [f'(a) + u(x)](x-a) \text{ for all } x \in G$$

and $\lim_{x\to a} u(x) = 0$.

Proof: Define

$$u(x) = \begin{cases} \dfrac{f(x)-f(a)}{x-a} - f'(a), & x \in G \setminus \{a\} \\ 0, & x = a. \end{cases}$$

Since f is differentiable at a, we have

$$\lim_{x\to a} u(x) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} - f'(a) = f'(a) - f'(a) = 0.$$

Therefore, the function u satisfies the conditions of the lemma. □

Theorem 4.1.5 — Chain rule. Let $f\colon G_1 \to \mathbb{R}$ and let $g\colon G_2 \to \mathbb{R}$, where G_1 and G_2 are two open subsets of $\mathbb{R}$ with $f(G_1) \subset G_2$. Suppose f is differentiable at a and g is differentiable at $f(a)$. Then the function $g \circ f$ is differentiable at a and

$$(g\circ f)'(a) = g'(f(a))f'(a).$$

Proof: Since f is differentiable at a, by Lemma 4.1.4, there exists a function u defined on G_1 with

$$f(x) - f(a) = [f'(a) + u(x)](x-a) \text{ for all } x \in G_1,$$

and $\lim_{x\to a} u(x) = 0$.

Similarly, since g is differentiable at $f(a)$, there exists a function v defined on G_2 with

$$g(t) - g(f(a)) = [g'(f(a)) + v(t)][t - f(a)] \text{ for all } t \in G_2, \tag{4.2}$$

and $\lim_{t\to f(a)} v(t) = 0$.

Applying (4.2) for $t = f(x)$, we have

$$g(f(x)) - g(f(a)) = [g'(f(a)) + v(f(x))][f(x) - f(a)].$$

Thus,

$$g(f(x)) - g(f(a)) = [g'(f(a)) + v(f(x))][f'(a) + u(x)](x-a) \text{ for all } x \in G_1.$$

This implies

$$\frac{g(f(x)) - g(f(a))}{x-a} = [g'(f(a)) + v(f(x))][f'(a) + u(x)] \text{ for all } x \in G_1 \setminus \{a\}.$$

By the continuity of f at a and the property of v, we have $\lim_{x\to a} v(f(x)) = 0$ and, hence,

$$\lim_{x\to a} \frac{g(f(x)) - g(f(a))}{x-a} = g'(f(a))f'(a).$$

The proof is now complete. □

■ **Example 4.1.3** Consider the function $h\colon \mathbb{R} \to \mathbb{R}$ given by $h(x) = (3x^4 + x + 7)^{15}$. Since $h(x)$ is a polynomial we could in principle compute $h'(x)$ by expanding the power and using Example 4.1.2. However, Theorem 4.1.5 provides a shorter way. Define $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = 3x^4 + x + 7$ and $g(x) = x^{15}$. Then $h = g \circ f$. Given $a \in \mathbb{R}$, it follows from Theorem 4.1.5 that

$$(g \circ f)'(a) = g'(f(a))f'(a) = 15(3a^4 + a + 7)^{14}(12a^3 + 1).$$

■ **Example 4.1.4** By iterating the Chain Rule, we can extended the result to the composition of more than two functions in a straightforward way. For example, given functions $f\colon G_1 \to \mathbb{R}$, $g\colon G_2 \to \mathbb{R}$, and $h\colon G_3 \to \mathbb{R}$ such that $f(G_1) \subset G_2$, $g(G_2) \subset G_3$, f is differentiable at a, g is differentiable at $f(a)$, and h is differentiable at $g(f(a))$, we obtain that $h \circ g \circ f$ is differentiable at a and $(h \circ g \circ f)'(a) = h'(g(f(a)))g'(f(a))f'(a)$.

Definition 4.1.2 Let G be an open set and let $f\colon G \to \mathbb{R}$ be a differentiable function. If the function $f'\colon G \to \mathbb{R}$ is also differentiable, we say that f is *twice differentiable* (on G). The second derivative of f is denoted by f'' or $f^{(2)}$. Thus, $f'' = (f')'$. Similarly, we say that f is three times differentiable if $f^{(2)}$ is differentiable, and $(f^{(2)})'$ is called the third derivative of f and is denoted by f''' or $f^{(3)}$. We can define in this way *n times differentiability* and the *nth* derivative of f for any positive integer n. As a convention, $f^{(0)} = f$.

Definition 4.1.3 Let I be an open interval in $\mathbb{R}$ and let $f\colon I \to \mathbb{R}$. The function f is said to be *continuously differentiable* if f is differentiable on I and f' is continuous on I. We denote by $C^1(I)$ the set of all continuously differentiable functions on I. If f is n times differentiable on I and the nth derivative is continuous, then f is called *n times continuously differentiable*. We denote by $C^n(I)$ the set of all n times continuously differentiable functions on I.

Exercises

4.1.1 Prove parts (a) and (b) of Theorem 4.1.3.

4.1.2 Compute the following derivatives directly from the definition. That is, do not use Theorem 4.1.3, but rather compute the appropriate limit directly (see Example 4.1.1).

(a) $f(x) = mx + b$ where $m, b \in \mathbb{R}$.

(b) $f(x) = \dfrac{1}{x}$ (here assume $x \neq 0$).

(c) $f(x) = \sqrt{x}$ (here assume $x > 0$)

4.1.3 Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^2, & \text{if } x > 0; \\ 0, & \text{if } x \leq 0. \end{cases}$$

(a) Prove that f is differentiable at 0. Find $f'(x)$ for all $x \in \mathbb{R}$.

(b) Is f' continuous? Is f' differentiable?

4.1.4 Let

$$f(x) = \begin{cases} x^\alpha, & \text{if } x > 0; \\ 0, & \text{if } x \leq 0. \end{cases}$$

(a) Determine the values of α for which f is continuous on $\mathbb{R}$.

(b) Determine the values of α for which f is differentiable on $\mathbb{R}$. In this case, find f'.

4.1.5 Use Theorems 4.1.3 and 4.1.5 to compute the derivatives of the following functions at the indicated points (see also Example 4.1.4). (Assume known that the function $\sin x$ is differentiable at all points and that its derivative is $\cos x$.)

(a) $f(x) = \dfrac{3x^4 + 7x}{2x^2 + 3}$ at $a = -1$.

(b) $f(x) = \sin^5(3x^2 + \frac{\pi}{2}x)$ at $a = \frac{\pi}{8}$

4.1.6 Determine the values of x at which each function is differentiable.

(a) $f(x) = \begin{cases} x\sin\dfrac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$

(b) $f(x) = \begin{cases} x^2\sin\dfrac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$

4.1.7 Determine if each of the following functions is differentiable at 0. Justify your answer.

(a) $f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q}; \\ x^3, & \text{if } x \notin \mathbb{Q}. \end{cases}$

(b) $f(x) = [x]\sin^2(\pi x)$.

(c) $f(x) = \cos(\sqrt{|x|})$.

(d) $f(x) = x|x|$.

4.1.8 Let f, g be differentiable at a. Find the following limits:

(a) $\lim_{x\to a} \dfrac{xf(a) - af(x)}{x - a}$.

(b) $\lim_{x\to a} \dfrac{f(x)g(a) - f(a)g(x)}{x - a}$.

4.1.9 Let G be an open subset of $\mathbb{R}$ and $a \in G$. Prove that if $f\colon G \to \mathbb{R}$ is Lipschitz continuous, then $g(x) = (f(x) - f(a))^2$ is differentiable at a.

4.1.10 ▷ Let f be differentiable at a and $f(a) > 0$. Find the following limit:

$$\lim_{n\to\infty} \left(\frac{f(a + \frac{1}{n})}{f(a)} \right)^n.$$

4.1.11 ▶ Consider the function

$$f(x) = \begin{cases} x^2\sin\dfrac{1}{x} + cx, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0, \end{cases}$$

where $0 < c < 1$.

(a) Prove that the function is differentiable on $\mathbb{R}$.

(b) Prove that for every $\alpha > 0$, the function f' changes its sign on $(-\alpha, \alpha)$.

4.1.12 Let f be differentiable at $x_0 \in (a,b)$ and let c be a constant. Prove that

(a) $\lim_{n\to\infty} n\left[f(x_0 + \frac{1}{n}) - f(x_0)\right] = f'(x_0)$.

(b) $\lim_{h\to 0} \dfrac{f(x_0 + ch) - f(x_0)}{h} = cf'(x_0)$.

4.1.13 Let f be differentiable at $x_0 \in (a,b)$ and let c be a constant. Find the limit

$$\lim_{h\to 0} \frac{f(x_0 + ch) - f(x_0 - ch)}{h}.$$

4.1.14 Prove that $f\colon \mathbb{R} \to \mathbb{R}$, given by $f(x) = |x|^3$, is in $C^2(\mathbb{R})$ but not in $C^3(\mathbb{R})$ (refer to Definition 4.1.3). (*Hint:* the key issue is differentiability at 0.)

4.2 THE MEAN VALUE THEOREM

In this section, we focus on the Mean Value Theorem, one of the most important tools of calculus and one of the most beautiful results of mathematical analysis. The Mean Value Theorem we study in this section was stated by the French mathematician Augustin Louis Cauchy (1789–1857), which follows from a simpler version called Rolle's Theorem.

An important application of differentiation is solving optimization problems. A simple method for identifying local extrema of a function was found by the French mathematician Pierre de Fermat (1601–1665). Fermat's method can also be used to prove Rolle's Theorem.

We start with some basic definitions of minima and maxima. Recall that for $a \in \mathbb{R}$ and $\delta > 0$, the sets $B(a;\delta)$, $B_+(a;\delta)$, and $B_-(a;\delta)$ denote the intervals $(a-\delta, a+\delta)$, $(a, a+\delta)$ and $(a-\delta, a)$, respectively.

Definition 4.2.1 Let D be a nonempty subset of $\mathbb{R}$ and let $f\colon D \to \mathbb{R}$. We say that f has a *local (or relative) minimum at* $a \in D$ if there exists $\delta > 0$ such that

$$f(x) \geq f(a) \text{ for all } x \in B(a;\delta) \cap D.$$

Similarly, we say that f has a *local (or relative) maximum at* $a \in D$ if there exists $\delta > 0$ such that

$$f(x) \leq f(a) \text{ for all } x \in B(a;\delta) \cap D.$$

In January 1638, Pierre de Fermat described his method for finding maxima and minima in a letter written to Marin Mersenne (1588–1648) who was considered as "the center of the world of science and mathematics during the first half of the 1600s." His method presented in the theorem below is now known as Fermat's Rule.

Theorem 4.2.1 — Fermat's Rule. Let I be an open interval and $f\colon I \to \mathbb{R}$. If f has a local minimum or maximum at $a \in I$ and f is differentiable at a, then $f'(a) = 0$.

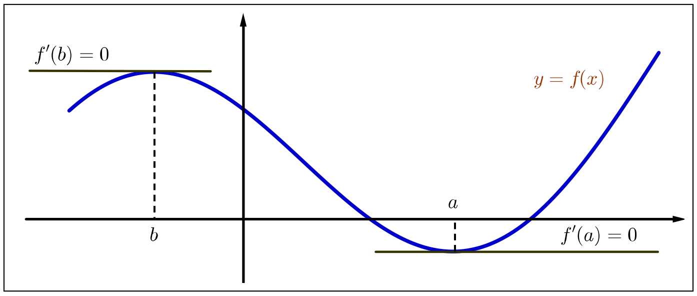

Figure 4.1: Illustration of Fermat's Rule.

Proof: Suppose f has a local minimum at a. Then there exists $\delta > 0$ sufficiently small such that

$$f(x) \geq f(a) \text{ for all } x \in B(a;\delta).$$

Since $B_+(a;\delta)$ is a subset of $B(a;\delta)$, we have

$$\frac{f(x)-f(a)}{x-a} \geq 0 \text{ for all } x \in B_+(a;\delta).$$

Taking into account the differentiability of f at a yields

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} \geq 0.$$

Similarly,

$$\frac{f(x)-f(a)}{x-a} \leq 0 \text{ for all } x \in B_-(a;\delta).$$

It follows that

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a^-} \frac{f(x)-f(a)}{x-a} \leq 0.$$

Therefore, $f'(a) = 0$. The proof is similar for the case where f has a local maximum at a. $\square$

Theorem 4.2.2 — Rolle's Theorem. Let $a,b \in \mathbb{R}$ with $a < b$ and $f\colon [a,b] \to \mathbb{R}$. Suppose f is continuous on $[a,b]$ and differentiable on (a,b) with $f(a) = f(b)$. Then there exists $c \in (a,b)$ such that

$$f'(c) = 0. \tag{4.3}$$

Proof: Since f is continuous on the compact set $[a,b]$, by the extreme value theorem (Theorem 3.4.2) there exist $\bar{x}_1 \in [a,b]$ and $\bar{x}_2 \in [a,b]$ such that

$$f(\bar{x}_1) = \min\{f(x) : x \in [a,b]\} \text{ and } f(\bar{x}_2) = \max\{f(x) : x \in [a,b]\}.$$

Then

$$f(\bar{x}_1) \leq f(x) \leq f(\bar{x}_2) \text{ for all } x \in [a,b]. \tag{4.4}$$

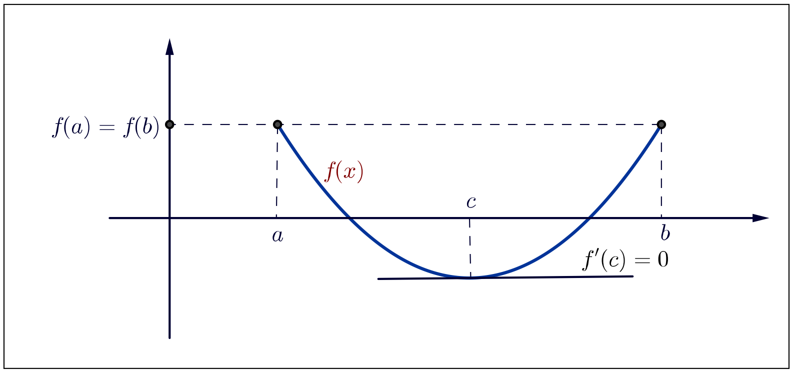

Figure 4.2: Illustration of Rolle's Theorem.

If $\bar{x}_1 \in (a,b)$ or $\bar{x}_2 \in (a,b)$, then f has a local minimum at $\bar{x}_1$ or f has a local maximum at $\bar{x}_2$. By Theorem 4.2.1, $f'(\bar{x}_1) = 0$ or $f'(\bar{x}_2) = 0$, and (4.3) holds with $c = \bar{x}_1$ or $c = \bar{x}_2$.

If both $\bar{x}_1$ and $\bar{x}_2$ are the endpoints of $[a,b]$, then $f(\bar{x}_1) = f(\bar{x}_2)$ because $f(a) = f(b)$. By (4.4), f is a constant function, so $f'(c) = 0$ for any $c \in (a,b)$. $\square$

We are now ready to use Rolle's Theorem to prove the Mean Value Theorem presented below.

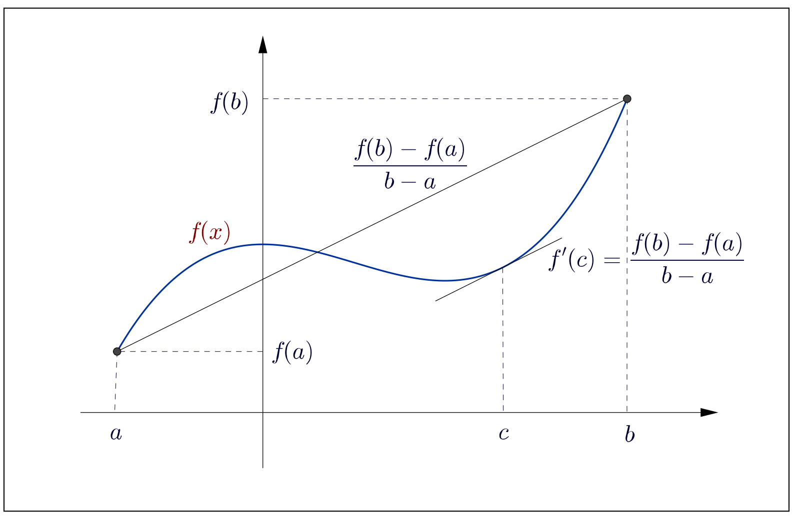

Figure 4.3: Illustration of the Mean Value Theorem.

Theorem 4.2.3 — Mean Value Theorem. Let $a,b \in \mathbb{R}$ with $a < b$ and $f\colon [a,b] \to \mathbb{R}$. Suppose f is continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{4.5}$$

Proof: The linear function whose graph goes through $(a, f(a))$ and $(b, f(b))$ is

$$g(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Define

$$h(x) = f(x) - g(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right] \text{ for } x \in [a,b].$$

Then $h(a) = h(b)$, and h satisfies the assumptions of Theorem 4.2.2. Thus, there exists $c \in (a,b)$ such that $h'(c) = 0$. Since

$$h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a},$$

it follows that

$$f'(c) - \frac{f(b)-f(a)}{b-a} = 0.$$

Thus, (4.5) holds. □

■ **Example 4.2.1** We show that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$.

Let $f(x) = \sin x$ for all $x \in \mathbb{R}$. Then $f'(x) = \cos x$. Now, fix $x \in \mathbb{R}$, $x > 0$. By the Mean Value Theorem applied to f on the interval $[0,x]$, there exists $c \in (0,x)$ such that

$$\frac{\sin x - \sin 0}{x - 0} = \cos c.$$

Therefore, $\dfrac{|\sin x|}{|x|} = |\cos c|$. Since $|\cos c| \leq 1$ we conclude $|\sin x| \leq |x|$ for all $x > 0$. Next suppose $x < 0$. Another application of the Mean Value Theorem shows there exists $c \in (x,0)$ such that

$$\frac{\sin 0 - \sin x}{0 - x} = \cos c.$$

Then, again, $\dfrac{|\sin x|}{|x|} = |\cos c| \leq 1$. It follows that $|\sin x| \leq |x|$ for $x < 0$. Since equality holds for $x = 0$, we conclude that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$.

■ **Example 4.2.2** We show that $\sqrt{1+4x} < (5+2x)/3$ for all $x > 2$.

Let $f(x) = \sqrt{1+4x}$ for all $x \geq 2$. Then

$$f'(x) = \frac{4}{2\sqrt{1+4x}} = \frac{2}{\sqrt{1+4x}}.$$

Now, fix $x \in \mathbb{R}$ such that $x > 2$. We apply the Mean Value Theorem to f on the interval $[2,x]$. Then, since $f(2) = 3$, there exists $c \in (2,x)$ such that

$$\sqrt{1+4x} - 3 = f'(c)(x-2).$$

Since $f'(2) = 2/3$ and $f'(c) < f'(2)$ for $c > 2$ we conclude that

$$\sqrt{1+4x} - 3 < \frac{2}{3}(x-2).$$

Rearranging terms provides the desired inequality.

A more general result which follows directly from the Mean Value Theorem is known as Cauchy's Theorem.

Theorem 4.2.4 — Cauchy's Theorem. Let $a, b \in \mathbb{R}$ with $a < b$. Suppose f and g are continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c). \tag{4.6}$$

Proof: Define

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x) \text{ for } x \in [a,b].$$

Then $h(a) = f(b)g(a) - f(a)g(b) = h(b)$, and h satisfies the assumptions of Theorem 4.2.2. Thus, there exists $c \in (a,b)$ such that $h'(c) = 0$. Since

$$h'(x) = [f(b) - f(a)]g'(x) - [g(b) - g(a)]f'(x),$$

this implies (4.6). □

The following theorem shows that the derivative of a differentiable function on $[a,b]$ satisfies the intermediate value property although the derivative function is not assumed to be continuous. To give the theorem in its greatest generality, we introduce a couple of definitions.

Definition 4.2.2 Let $a, b \in \mathbb{R}$, $a < b$, and $f \colon [a,b] \to \mathbb{R}$. If the limit

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$$

exists, we say that f has a *right derivative* at a and write

$$f'_+(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}.$$

If the limit

$$\lim_{x \to b^-} \frac{f(x) - f(b)}{x - b}$$

exists, we say that f has a *left derivative* at b and write

$$f'_-(b) = \lim_{x \to b^-} \frac{f(x) - f(b)}{x - b}.$$

We will say that f is differentiable on $[a,b]$ if $f'(x)$ exists for each $x \in (a,b)$ and, in addition, both $f'_+(a)$ and $f'_-(b)$ exist.

Theorem 4.2.5 — Intermediate Value Theorem for Derivatives. Let $a, b \in \mathbb{R}$ with $a < b$. Suppose f is differentiable on $[a,b]$ and

$$f'_+(a) < \lambda < f'_-(b).$$

Then there exists $c \in (a,b)$ such that

$$f'(c) = \lambda.$$

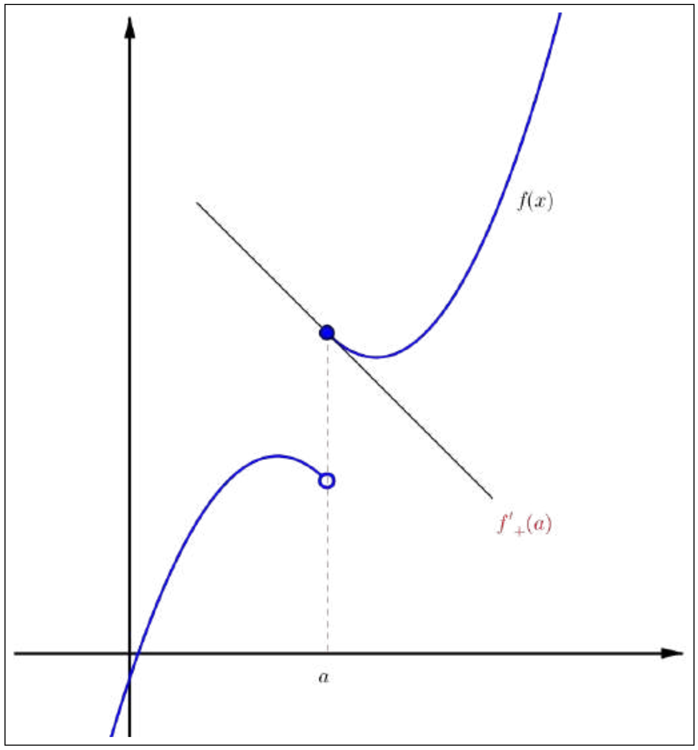

Figure 4.4: Right derivative.

Proof: Define the function $g\colon [a,b] \to \mathbb{R}$ by

$$g(x) = f(x) - \lambda x.$$

Then g is differentiable on $[a,b]$ and

$$g'_+(a) < 0 < g'_-(b).$$

Thus,

$$\lim_{x\to a^+} \frac{g(x)-g(a)}{x-a} < 0.$$

It follows that there exists $\delta_1 > 0$ such that

$$g(x) < g(a) \text{ for all } x \in (a, a+\delta_1) \cap [a,b].$$

Similarly, there exists $\delta_2 > 0$ such that

$$g(x) < g(b) \text{ for all } x \in (b-\delta_2, b) \cap [a,b].$$

Since g is continuous on $[a,b]$, it attains its minimum at a point $c \in [a,b]$. From the observations above, it follows that $c \in (a,b)$. This implies $g'(c) = 0$ or, equivalently, that $f'(c) = \lambda$. $\square$

Remark 4.2.6 The same conclusion follows if $f'_+(a) > \lambda > f'_-(b)$.

Exercises

4.2.1 ▷ Let f and g be differentiable at x_0. Suppose $f(x_0) = g(x_0)$ and

$$f(x) \le g(x) \text{ for all } x \in \mathbb{R}.$$

Prove that $f'(x_0) = g'(x_0)$.

4.2.2 Prove the following inequalities using the Mean Value Theorem.

(a) $\sqrt{1+x} < 1 + \frac{1}{2}x$ for $x > 0$.

(b) $e^x > 1 + x$, for $x > 0$. (Assume known that the derivative of e^x is itself.)

(c) $\dfrac{x-1}{x} < \ln x < x - 1$, for $x > 1$. (Assume known that the derivative of $\ln x$ is $1/x$.)

4.2.3 ▶ Prove that $|\sin(x) - \sin(y)| \le |x - y|$ for all $x, y \in \mathbb{R}$.

4.2.4 ▷ Let n be a positive integer and let $a_k, b_k \in \mathbb{R}$ for $k = 1, \ldots, n$. Prove that the equation

$$x + \sum_{k=1}^{n} (a_k \sin kx + b_k \cos kx) = 0$$

has a solution on $(-\pi, \pi)$.

4.2.5 ▷ Let f and g be differentiable functions on $[a,b]$. Suppose $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in [a,b]$. Prove that there exists $c \in (a,b)$ such that

$$\frac{1}{g(b)-g(a)} \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = \frac{1}{g'(c)} \begin{vmatrix} f(c) & g(c) \\ f'(c) & g'(c) \end{vmatrix},$$

where the bars denote determinants of the two-by-two matrices.

4.2.6 ▷Let n be a fixed positive integer.

(a) Suppose $a_1, a_2, \ldots, a_n$ satisfy

$$a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n} = 0.$$

Prove that the equation

$$a_1 + a_2 x + a_3 x^2 + \cdots + a_n x^{n-1} = 0$$

has a solution in $(0,1)$.

(b) Suppose $a_0, a_1, \ldots, a_n$ satisfy

$$\sum_{k=0}^{n} \frac{a_k}{2k+1} = 0.$$

Prove that the equation

$$\sum_{k=0}^{n} a_k \cos(2k+1)x = 0$$

has a solution on $(0, \frac{\pi}{2})$.

4.2.7 Let $f\colon [0,\infty) \to \mathbb{R}$ be a differentiable function. Prove that if both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ exist, then $\lim_{x\to\infty} f'(x) = 0$

4.2.8 ▷ Let $f\colon [0,\infty) \to \mathbb{R}$ be a differentiable function.

(a) Show that if $\lim_{x\to\infty} f'(x) = a$, then $\lim_{x\to\infty} \dfrac{f(x)}{x} = a$.

(b) Show that if $\lim_{x\to\infty} f'(x) = \infty$, then $\lim_{x\to\infty} \dfrac{f(x)}{x} = \infty$.

(c) Are the converses in part (a) and part (b) true?

4.3 SOME APPLICATIONS OF THE MEAN VALUE THEOREM

In this section, we assume that $a, b \in \mathbb{R}$ and $a < b$. In the proposition below, we show that it is possible to use the derivative to determine whether a function is constant. The proof is based on the Mean Value Theorem.

Proposition 4.3.1 Let f be continuous on $[a,b]$ and differentiable on (a,b). If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on $[a,b]$.

Proof: Suppose by contradiction that f is not constant on $[a,b]$. Then there exist a_1 and b_1 such that $a \le a_1 < b_1 \le b$ and $f(a_1) \ne f(b_1)$. By Theorem 4.2.3, there exists $c \in (a_1, b_1)$ such that

$$f'(c) = \frac{f(b_1) - f(a_1)}{b_1 - a_1} \ne 0,$$

which is a contradiction. □

The next application of the Mean Value Theorem concerns developing simple criteria for monotonicity of real-valued functions based on the derivative.

Proposition 4.3.2 Let f be differentiable on (a,b).

(i) If $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing on (a,b).
(ii) If $f'(x) < 0$ for all $x \in (a,b)$, then f is strictly decreasing on (a,b).

Proof: Let us prove (i). Fix any $x_1, x_2 \in (a,b)$ with $x_1 < x_2$. By Theorem 4.2.3, there exists $c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0.$$

This implies $f(x_1) < f(x_2)$. Therefore, f is strictly increasing on (a,b). The proof of (ii) is similar. □

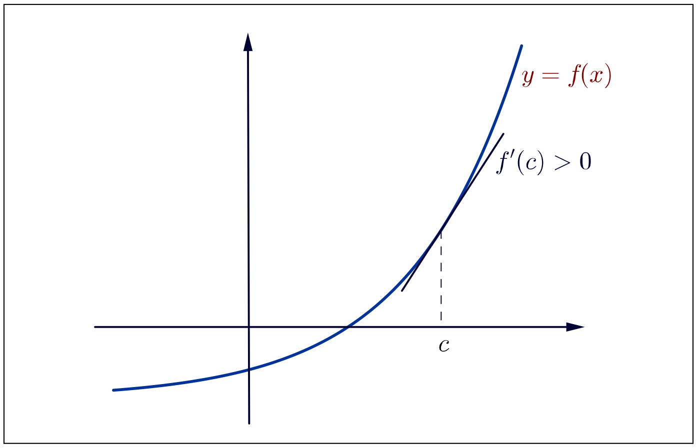

Figure 4.5: Strictly Increasing Function.

■ **Example 4.3.1** Let $n \in \mathbb{N}$ and $f\colon [0,\infty) \to \mathbb{R}$ be given by $f(x) = x^n$. Then $f'(x) = nx^{n-1}$. Therefore, $f'(x) > 0$ for all $x > 0$ and, so, f is strictly increasing. In particular, this shows that every positive real number has exactly one n-th root (refer to Example 3.4.2).

Theorem 4.3.3 — Inverse Function Theorem. Suppose f is differentiable on $I = (a,b)$ and $f'(x) \neq 0$ for all $x \in (a,b)$. Then f is one-to-one, $f(I)$ is an open interval, and the inverse function $f^{-1}\colon f(I) \to I$ is differentiable. Moreover,

$$(f^{-1})'(y) = \frac{1}{f'(x)}, \tag{4.7}$$

where $f(x) = y$.

Proof: It follows from Theorem 4.2.5 that

$$f'(x) > 0 \text{ for all } x \in (a,b), \text{ or } f'(x) < 0 \text{ for all } x \in (a,b).$$

Suppose $f'(x) > 0$ for all $x \in (a,b)$. Then f is strictly increasing on this interval and, hence, it is one-to-one. It follows from Theorem 3.4.10 and Remark 3.4.11 that $f(I)$ is an open interval and f^{-1} is continuous on $f(I)$.

It remains to prove the differentiability of the inverse function f^{-1} and the representation of its derivative (4.7). Fix any $\bar{y} \in f(I)$ with $\bar{y} = f(\bar{x})$. Let $g = f^{-1}$. We will show that

$$\lim_{y\to\bar{y}} \frac{g(y) - g(\bar{y})}{y - \bar{y}} = \frac{1}{f'(\bar{x})}.$$

Fix any sequence $\{y_k\}$ in $f(I)$ that converges to $\bar{y}$ and $y_k \neq \bar{y}$ for every k. For each y_k, there exists $x_k \in I$ such that $f(x_k) = y_k$. That is, $g(y_k) = x_k$ for all k. It follows from the continuity of g that $\{x_k\}$ converges to $\bar{x}$. Then

$$\begin{aligned}
\lim_{k\to\infty} \frac{g(y_k) - g(\bar{y})}{y_k - \bar{y}} &= \lim_{k\to\infty} \frac{x_k - \bar{x}}{f(x_k) - f(\bar{x})} \\
&= \lim_{k\to\infty} \frac{1}{\dfrac{f(x_k) - f(\bar{x})}{x_k - \bar{x}}} = \frac{1}{f'(\bar{x})}.
\end{aligned}$$

The proof is now complete. □

■ **Example 4.3.2** Let $n \in \mathbb{N}$ and consider the function $f\colon (0,\infty) \to \mathbb{R}$ given by $f(x) = x^n$. Then f is differentiable and $f'(x) = nx^{n-1} \neq 0$ for all $x \in (0,\infty)$. It is also clear that $f((0,\infty)) = (0,\infty)$. It follows from the Inverse Function Theorem that $f^{-1} : (0,\infty) \to (0,\infty)$ is differentiable and given $y \in (0,\infty)$

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{n(f^{-1}(y))^{n-1}}.$$

Given $y > 0$, the value $f^{-1}(y)$ is the unique positive real number whose n-th power is y. We call $f^{-1}(y)$ the (positive) n-th root of y and denote it by $\sqrt[n]{y}$. We also obtain the formula

$$(f^{-1})'(y) = \frac{1}{n(\sqrt[n]{y})^{n-1}}.$$

Exercises

4.3.1 (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable. Prove that if $f'(x)$ is bounded, then f is Lipschitz continuous and, in particular, uniformly continuous.

(b) Give an example of a function $f\colon (0,\infty) \to \mathbb{R}$ which is differentiable and uniformly continuous but such that $f'(x)$ is not bounded.

4.3.2 ▶ Let $f\colon \mathbb{R} \to \mathbb{R}$. Suppose there exist $\ell \geq 0$ and $\alpha > 0$ such that

$$|f(u) - f(v)| \leq \ell |u - v|^{\alpha} \text{ for all } u, v \in \mathbb{R}. \tag{4.8}$$

(a) Prove that f is uniformly continuous on $\mathbb{R}$.

(b) Prove that if $\alpha > 1$, then f is a constant function.

(c) Find a nondifferentiable function that satisfies the condition above for $\alpha = 1$.

4.3.3 ▷ Let f and g be differentiable functions on $\mathbb{R}$ such that $f(x_0) = g(x_0)$ and

$$f'(x) \leq g'(x) \text{ for all } x \geq x_0.$$

Prove that

$$f(x) \leq g(x) \text{ for all } x \geq x_0.$$

4.3.4 Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be differentiable functions satisfying

(a) $f(0) = g(0) = 1$

(b) $f(x) > 0$, $g(x) > 0$ and $\dfrac{f'(x)}{f(x)} > \dfrac{g'(x)}{g(x)}$ for all x.

Prove that

$$\frac{f(1)}{g(1)} > 1 > \frac{g(1)}{f(1)}.$$

4.3.5 ▷ Let f be twice differentiable on an open interval I. Suppose that there exist $a,b,c \in I$ with $a < b < c$ such that $f(a) < f(b)$ and $f(b) > f(c)$. Prove that there exists $d \in (a,c)$ such that $f''(d) < 0$.

4.3.6 ▷ Prove that the function f defined in Exercise 4.1.11 is not monotone on any open interval containing 0.

4.4 L'HOSPITAL'S RULE

We now prove a result that allows us to compute various limits by calculating a related limit involving derivatives. All four theorems in this section are known as **l'Hospital's Rule**.

For this section, we assume $a,b \in \mathbb{R}$ with $a < b$.

Theorem 4.4.1 Suppose f and g are continuous on $[a,b]$ and differentiable on (a,b). Suppose $f(\bar{x}) = g(\bar{x}) = 0$, where $\bar{x} \in [a,b]$. Suppose further that there exists $\delta > 0$ such that $g'(x) \neq 0$ for all $x \in B(\bar{x};\delta) \cap [a,b]$, $x \neq \bar{x}$.

If

$$\lim_{x\to\bar{x}} \frac{f'(x)}{g'(x)} = \ell,$$

then

$$\lim_{x\to\bar{x}} \frac{f(x)}{g(x)} = \ell. \tag{4.9}$$

Proof: Let $\{x_k\}$ be a sequence in $[a,b]$ that converges to $\bar{x}$ and such that $x_k \neq \bar{x}$ for every k. By Theorem 4.2.4, for each k, there exists a sequence $\{c_k\}$, with c_k between x_k and $\bar{x}$, such that

$$[f(x_k) - f(\bar{x})]g'(c_k) = [g(x_k) - g(\bar{x})]f'(c_k).$$

Since $f(\bar{x}) = g(\bar{x}) = 0$, and $g'(c_k) \neq 0$ for sufficiently large k, we have

$$\frac{f(x_k)}{g(x_k)} = \frac{f'(c_k)}{g'(c_k)}.$$

Under the assumptions that $g'(x) \neq 0$ for x near $\bar{x}$ and $g(\bar{x}) = 0$, we also have $g(x_k) \neq 0$ for sufficiently large k. By the squeeze theorem (Theorem 2.1.6), $\{c_k\}$ converges to $\bar{x}$. Thus,

$$\lim_{k\to\infty} \frac{f(x_k)}{g(x_k)} = \lim_{k\to\infty} \frac{f'(c_k)}{g'(c_k)} = \lim_{x\to\bar{x}} \frac{f'(x)}{g'(x)} = \ell.$$

Therefore, (4.9) follows from Theorem 3.1.2. □

■ **Example 4.4.1** We will use Theorem 4.4.1 to show that

$$\lim_{x\to 0} \frac{2x + \sin x}{x^2 + 3x} = 1.$$

First we observe that the conditions of Theorem 4.4.1 hold. Here $f(x) = 2x + \sin x$, $g(x) = x^2 + 3x$, and $\bar{x} = 0$. We may take $[a,b] = [-1,1]$, for example, so that f and g are continuous on $[a,b]$ and differentiable on (a,b) and, furthermore, $\frac{f(x)}{g(x)}$ is well defined on $[a,b] \setminus \{\bar{x}\}$. Moreover, taking $\delta = 7/3$, we get $g'(x) = 2x + 3 \neq 0$ for $x \in B(\bar{x};\delta) \cap [a,b]$. Finally we calculate the limit of the quotient of derivatives using Theorem 3.2.1 to get

$$\lim_{x\to\bar{x}} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{2+\cos x}{2x+3} = \frac{\lim_{x\to 0} 2 + \lim_{x\to 0} \cos x}{\lim_{x\to 0} 2x+3} = \frac{2+1}{3} = 1.$$

It now follows from Theorem 4.4.1 that $\lim_{x\to 0} \frac{2x+\sin x}{x^2+3x} = 1$ as we wanted to show.

■ **Example 4.4.2** We will apply L'Hospital's rule to determine the limit

$$\lim_{x\to 1} \frac{3x^3 - 2x^2 + 4x - 5}{4x^4 - 2x - 2}.$$

Here $f(x) = 3x^3 - 2x^2 + 4x - 5$ and $g(x) = 4x^4 - 2x - 2$. Thus $f(1) = g(1) = 0$. Moreover, $f'(x) = 9x^2 - 4x + 4$ and $g'(x) = 16x^3 - 2$. Since $g'(1) = 14 \neq 0$ and g' is continuous we have $g'(x) \neq 0$ for x near 1. Now,

$$\lim_{x\to 1} \frac{9x^2 - 4x + 4}{16x^3 - 2} = \frac{9}{14}.$$

Thus, the desired limit is $\frac{9}{14}$ as well.

■ **Example 4.4.3** If the derivatives of the functions f and g themselves satisfy the assumptions of Theorem 4.4.1 we may apply L'Hospital's rule to determine first the limit of $f'(x)/g'(x)$ and then apply the rule again to determine the original limit.

Consider the limit

$$\lim_{x\to 0} \frac{x^2}{1-\cos x}.$$

Here $f(x) = x^2$ and $g(x) = 1 - \cos x$ so both functions and all its derivatives are continuous. Now $g'(x) = \sin x$ and, so, $g'(x) \neq 0$ for x near zero, $x \neq 0$. Also, $f'(0) = 0 = g'(0)$ and $g''(x) = \cos x \neq 0$ for x near 0. Moreover,

$$\lim_{x\to 0} \frac{f''(x)}{g''(x)} = \lim_{x\to 0} \frac{2}{\cos x} = 2.$$

By L'Hospital's rule we get

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f''(x)}{g''(x)} = \lim_{x\to 0} \frac{2}{\cos x} = 2.$$

Applying L'Hospital's rule one more time we get

$$\lim_{x\to 0} \frac{x^2}{1-\cos x} = \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = 2.$$

■ **Example 4.4.4** Let $g(x) = x + 3x^2$ and let $f: \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Now consider the limit

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{x + 3x^2}.$$

Using the derivative rules at $x \neq 0$ and the definition of derivative at $x = 0$ we can see that f is differentiable and

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0, \end{cases}$$

However, f' is not continuous at 0 (since $\lim_{x \to 0} f'(x)$ does not exist) and, hence, L'Hospital's rule cannot be applied in this case.

On the other hand $\lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{x + 3x^2}$ does exist as we can see from

$$\lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{x + 3x^2} = \lim_{x \to 0} \frac{x \sin \frac{1}{x}}{1 + 3x} = \frac{\lim_{x \to 0} x \sin \frac{1}{x}}{\lim_{x \to 0}(1 + 3x)} = 0.$$

Theorem 4.4.2 Let $a, b \in \mathbb{R}$, $a < b$, and $\bar{x} \in (a, b)$. Suppose $f, g: (a, b) \setminus \{\bar{x}\} \to \mathbb{R}$ are differentiable on $(a, b) \setminus \{\bar{x}\}$ and assume $\lim_{x \to \bar{x}} f(x) = \lim_{x \to \bar{x}} g(x) = \infty$. Suppose further that there exists $\delta > 0$ such that $g'(x) \neq 0$ for all $x \in B(\bar{x}; \delta) \cap (a, b)$, $x \neq \bar{x}$.

If $\ell \in \mathbb{R}$ and

$$\lim_{x \to \bar{x}} \frac{f'(x)}{g'(x)} = \ell, \tag{4.10}$$

then

$$\lim_{x \to \bar{x}} \frac{f(x)}{g(x)} = \ell. \tag{4.11}$$

Proof: Since $\lim_{x \to \bar{x}} f(x) = \lim_{x \to \bar{x}} g(x) = \infty$, choosing a smaller positive δ if necessary, we can assume that $f(x) \neq 0$ and $g(x) \neq 0$ for all $x \in B(\bar{x}; \delta) \cap (a, b)$.

We will show that $\lim_{x \to \bar{x}^+} \dfrac{f(x)}{g(x)} = \ell$. The proof that $\lim_{x \to \bar{x}^-} \dfrac{f(x)}{g(x)} = \ell$ is completely analogous.

Fix any $\varepsilon > 0$. We need to find $\delta_0 > 0$ such that $|f(x)/g(x) - \ell| < \varepsilon$ whenever $x \in B_+(\bar{x}; \delta_0) \cap (a, b)$.

From (4.10), one can choose $K > 0$ and a positive $\delta_1 < \delta$ such that

$$\left| \frac{f'(x)}{g'(x)} \right| \leq K \text{ and } \left| \frac{f'(x)}{g'(x)} - \ell \right| < \frac{\varepsilon}{2} \tag{4.12}$$

whenever $x \in B(\bar{x}; \delta_1) \cap (a, b)$, $x \neq \bar{x}$.

Fix $\alpha \in B_+(\bar{x}; \delta_1) \cap (a, b)$ (in particular, $\alpha > \bar{x}$). Since $\lim_{x \to \bar{x}} f(x) = \infty$, we can find $\delta_2 > 0$ such that $\delta_2 < \min\{\delta_1, \alpha - \bar{x}\}$ and $f(x) \neq f(\alpha)$ for $x \in B_+(\bar{x}; \delta_2) \cap (a, b) = B_+(\bar{x}; \delta_2)$. Moreover, for such x, since $g'(z) \neq 0$ if $x < z < \alpha$, Rolle's theorem (Theorem 4.2.2) guarantees that $g(x) \neq g(\alpha)$. Therefore, for all $x \in B_+(\bar{x}; \delta_2)$ we can write,

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(\alpha)}{g(x) - g(\alpha)} \frac{1 - \dfrac{g(\alpha)}{g(x)}}{1 - \dfrac{f(\alpha)}{f(x)}}.$$

Now, define

$$H_\alpha(x) = \frac{1 - \dfrac{g(\alpha)}{g(x)}}{1 - \dfrac{f(\alpha)}{f(x)}} \text{ for } x \in B_+(\bar{x}; \delta_2).$$

Since $\lim_{x\to\bar{x}} f(x) = \lim_{x\to\bar{x}} g(x) = \infty$, we have that $\lim_{x\to\bar{x}^+} H_\alpha(x) = 1$. Thus, there exists a positive $\gamma < \delta_2$ such that

$$|H_\alpha(x) - 1| < \frac{\varepsilon}{2K} \text{ whenever } x \in B_+(\bar{x}; \gamma).$$

For any $x \in B_+(\bar{x}; \gamma)$, applying Theorem 4.2.4 on the interval $[x, \alpha]$, we can write $[f(x) - f(\alpha)]g'(c) = [g(x) - g(\alpha)]f'(c)$ for some $c \in (x, \alpha)$ (note that, in particular, $c \in B(\bar{x} : \delta_1) \cap (a, b)$). For such c we get

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} H_\alpha(x).$$

Since $c \in B(\bar{x} : \delta_1) \cap (a, b)$, applying (4.12) we get that, for $x \in B_+(\bar{x}; \gamma) = B_+(\bar{x}; \gamma) \cap (a, b)$,

$$\begin{aligned}
\left| \frac{f(x)}{g(x)} - \ell \right| &= \left| \frac{f'(c)}{g'(c)} H_\alpha(x) - \ell \right| \\
&= \left| \frac{f'(c)}{g'(c)} (H_\alpha(x) - 1) + \frac{f'(c)}{g'(c)} - \ell \right| \\
&\leq \left| \frac{f'(c)}{g'(c)} \right| |H_\alpha(x) - 1| + \left| \frac{f'(c)}{g'(c)} - \ell \right| \\
&< K \frac{\varepsilon}{2K} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Setting $\delta_0 = \gamma$ completes the proof. □

■ **Example 4.4.5** Consider the limit

$$\lim_{x\to 0} \frac{\ln x^2}{1 + \frac{1}{\sqrt[3]{x^2}}}.$$

Here $f(x) = \ln x^2$, $g(x) = 1 + \frac{1}{\sqrt[3]{x^2}}$, $\bar{x} = 0$, and we may take as (a, b) any open inteval containing 0. Clearly f and g satisfy the differentiability assumptions and $g'(x) \neq 0$ for all $x \neq 0$. Moreover, $\lim_{x\to\bar{x}} f(x) = \lim_{x\to\bar{x}} g(x) = \infty$. We analyze the quotient of the derivatives. We have

$$\lim_{x\to 0} \frac{2/x}{-\frac{2}{3}\frac{1}{\sqrt[3]{x^5}}} = \lim_{x\to 0} -3\frac{\sqrt[3]{x^5}}{x} = \lim_{x\to 0} -3\sqrt[3]{x^2} = 0.$$

It now follows from Theorem 4.4.2 that

$$\lim_{x\to 0}\frac{\ln x^2}{1+\frac{1}{\sqrt[3]{x^2}}}=0.$$

Remark 4.4.3 The proofs of Theorem 4.4.1 and Theorem 4.4.2 show that the results in these theorems can be applied for left-hand and right-hand limits. Moreover, the results can also be modified to include the case when $\bar{x}$ is an endpoint of the domain of the functions f and g.

The following theorem can be proved following the method in the proof of Theorem 4.4.1.

Theorem 4.4.4 Let f and g be differentiable on (a,∞). Suppose $g'(x)\neq 0$ for all $x\in(a,\infty)$ and

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty} g(x)=0.$$

If $\ell\in\mathbb{R}$ and

$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)}=\ell,$$

then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\ell.$$

■ **Example 4.4.6** Consider the limit

$$\lim_{x\to\infty}\frac{1}{x(\frac{\pi}{2}-\arctan x)}.$$

Writing the quotient in the form $\frac{1/x}{\frac{\pi}{2}-\arctan x}$ we can apply Theorem 4.4.4. We now compute the limit of the quotient of the derivatives

$$\lim_{x\to\infty}\frac{-1/x^2}{-\frac{1}{x^2+1}}=\lim_{x\to\infty}\frac{x^2+1}{x^2}=1.$$

In view of Theorem 4.4.4 the desired limit is also 1.

The following theorem can be proved following the method in the proof of Theorem 4.4.2.

Theorem 4.4.5 Let f and g be differentiable on (a,∞). Suppose $g'(x)\neq 0$ for all $x\in(a,\infty)$ and

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty} g(x)=\infty.$$

If $\ell\in\mathbb{R}$ and

$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)}=\ell,$$

then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\ell.$$

■ **Example 4.4.7** Consider the limit

$$\lim_{x\to\infty}\frac{\ln x}{x}.$$

Clearly the functions $f(x)=\ln x$ and $g(x)=x$ satisfy the conditions of Theorem 4.4.5. We have

$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)}=\lim_{x\to\infty}\frac{1/x}{1}=0$$

It follows from Theorem 4.4.5 that $\lim_{x\to\infty}\frac{\ln x}{x}=0$.

Exercises

4.4.1 Use L'Hospital's Rule to find the following limits (you may assume known all the relevant derivatives from calculus):

(a) $\lim_{x\to -2}\frac{x^3-4x}{3x^2+5x-2}$.

(b) $\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x\cos x}$.

(c) $\lim_{x\to 1}\frac{x-1}{\sqrt{x+1}-\sqrt{2}}$.

(d) $\lim_{x\to 0}\frac{e^x-e^{-x}}{\ln(1+x)}$.

(e) $\lim_{x\to 1}\frac{\ln x}{\sin(\pi x)}$.

4.4.2 For the problems below use L'Hospital's rule as many times as appropriate to determine the limits.

(a) $\lim_{x\to 0}\frac{1-\cos 2x}{x\sin x}$.

(b) $\lim_{x\to 0}\frac{(x-\frac{\pi}{2})^2}{1-\sin x}$.

(c) $\lim_{x\to 0}\frac{x-\arctan x}{x^3}$.

(d) $\lim_{x\to 0}\frac{x-\sin x}{x-\tan x}$.

4.4.3 Use the relevant version of L'Hospital's rule to compute each of the following limits.

(a) $\lim_{x\to\infty}\frac{3x^2+2x+7}{4x^2-6x+1}$.

(b) $\lim_{x\to 0^+}\frac{-\ln x}{\cot x}$.

(c) $\lim_{x\to\infty} \frac{\frac{\pi}{2} - \arctan x}{\ln(1+\frac{1}{x})}$.

(d) $\lim_{x\to\infty} \sqrt{x}e^{-x}$. (*Hint:* first rewrite as a quotient.)

4.4.4 Prove that the following functions are differentiable at 1 and -1.

(a) $f(x) = \begin{cases} x^2 e^{-x^2}, & \text{if } |x| \le 1; \\ \frac{1}{e}, & \text{if } |x| > 1. \end{cases}$

(b) $f(x) = \begin{cases} \arctan x, & \text{if } |x| \le 1; \\ \frac{\pi}{4}\operatorname{sign} x + \frac{x-1}{2}, & \text{if } |x| > 1. \end{cases}$

4.4.5 ▷ Let $P(x)$ be a polynomial. Prove that

$$\lim_{x\to\infty} P(x)e^{-x} = 0.$$

4.4.6 ▶ Consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Prove that $f \in C^n(\mathbb{R})$ for every $n \in \mathbb{N}$.

4.5 TAYLOR'S THEOREM

In this section, we prove a result that lets us approximate differentiable functions by polynomials.

Theorem 4.5.1 — Taylor's Theorem. Let n be a positive integer. Suppose $f\colon [a,b] \to \mathbb{R}$ is a function such that $f^{(n)}$ is continuous on $[a,b]$, and $f^{(n+1)}(x)$ exists for all $x \in (a,b)$. Let $\bar{x} \in [a,b]$. Then for any $x \in [a,b]$ with $x \neq \bar{x}$, there exists a number c in between $\bar{x}$ and x such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-\bar{x})^{n+1},$$

where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(\bar{x})}{k!}(x-\bar{x})^k.$$

Proof: Let $\bar{x}$ be as in the statement and let us fix $x \neq \bar{x}$. Since $x - \bar{x} \neq 0$, there exists a number $\lambda \in \mathbb{R}$ such that

$$f(x) = P_n(x) + \frac{\lambda}{(n+1)!}(x-\bar{x})^{n+1}.$$

We will now show that

$$\lambda = f^{(n+1)}(c),$$

for some c in between $\bar{x}$ and x.

Consider the function

$$g(t) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k - \frac{\lambda}{(n+1)!}(x-t)^{n+1}.$$

Then

$$g(\bar{x}) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(\bar{x})}{k!}(x-\bar{x})^k - \frac{\lambda}{(n+1)!}(x-\bar{x})^{n+1} = f(x) - P_n(x) - \frac{\lambda}{(n+1)!}(x-\bar{x})^{n+1} = 0.$$

and

$$g(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(x-x)^k - \frac{\lambda}{(n+1)!}(x-x)^{n+1} = f(x) - f(x) = 0.$$

By Rolle's theorem, there exists c in between $\bar{x}$ and x such that $g'(c) = 0$. Taking the derivative of g (keeping in mind that x is fixed and the independent variable is t) and using the product rule for derivatives, we have

$$\begin{aligned} g'(c) &= -f'(c) + \sum_{k=1}^{n} \left(-\frac{f^{(k+1)}(c)}{k!}(x-c)^k + \frac{f^{(k)}(c)}{(k-1)!}(x-c)^{k-1} \right) + \frac{\lambda}{n!}(x-c)^n \\ &= \frac{\lambda}{n!}(x-c)^n - \frac{1}{n!} f^{(n+1)}(c)(x-c)^n \\ &= 0. \end{aligned}$$

This implies $\lambda = f^{(n+1)}(c)$. The proof is now complete. □

The polynomial $P_n(x)$ given in the theorem is called the *n-th Taylor polynomial of f at $\bar{x}$*.

Remark 4.5.2 The conclusion of Taylor's theorem still holds true if $x = \bar{x}$. In this case, $c = x = \bar{x}$.

■ **Example 4.5.1** We will use Taylor's theorem to estimate the error in approximating the function $f(x) = \sin x$ with it 3rd Taylor polynomial at $\bar{x} = 0$ on the interval $[-\pi/2, \pi/2]$. Since $f'(x) = \cos x$, $f''(x) = -\sin x$ and $f'''(x) = -\cos x$, a direct calculation shows that

$$P_3(x) = x - \frac{x^3}{3!}.$$

More over, for any $c \in \mathbb{R}$ we have $|f^{(4)}(c)| = |\sin c| \leq 1$. Therefore, for $x \in [-\pi/2, \pi/2]$ we get (for some c between x and 0),

$$|\sin x - P_3(x)| = \frac{|f^{(4)}(c)|}{4!}|x| \leq \frac{\pi/2}{4!} \leq 0.066.$$

Theorem 4.5.3 Let n be an even positive integer. Suppose $f^{(n)}$ exists and continuous on (a,b). Let $\bar{x} \in (a,b)$ satisfy

$$f'(\bar{x}) = \ldots = f^{(n-1)}(\bar{x}) = 0 \text{ and } f^{(n)}(\bar{x}) \neq 0.$$

The following hold:

(a) $f^{(n)}(\bar{x}) > 0$ if and only if f has a local minimum at $\bar{x}$.
(b) $f^{(n)}(\bar{x}) < 0$ if and only if f has a local maximum at $\bar{x}$.

Proof: We will prove (a). Suppose $f^{(n)}(\bar{x}) > 0$. Since $f^{(n)}(\bar{x}) > 0$ and $f^{(n)}$ is continuous at $\bar{x}$, there exists $\delta > 0$ such that

$$f^{(n)}(t) > 0 \text{ for all } t \in B(\bar{x};\delta) \subset (a,b).$$

Fix any $x \in B(\bar{x};\delta)$. By Taylor's theorem and the given assumption, there exists c in between $\bar{x}$ and x such that

$$f(x) = f(\bar{x}) + \frac{f^{(n)}(c)}{n!}(x-\bar{x})^n.$$

Since n is even and $c \in B(\bar{x};\delta)$, we have $f(x) \geq f(\bar{x})$. Thus, f has a local minimum at $\bar{x}$.

Now, for the converse, suppose that f has a local minimum at $\bar{x}$. Then there exists $\delta > 0$ such that

$$f(x) \geq f(\bar{x}) \text{ for all } x \in B(\bar{x};\delta) \subset (a,b).$$

Fix a sequence $\{x_k\}$ in (a,b) that converges to $\bar{x}$ with $x_k \neq \bar{x}$ for every k. By Taylor's theorem, there exists a sequence $\{c_k\}$, with c_k between x_k and $\bar{x}$ for each k, such that

$$f(x_k) = f(\bar{x}) + \frac{f^{(n)}(c_k)}{n!}(x_k-\bar{x})^n.$$

Since $x_k \in B(\bar{x};\delta)$ for sufficiently large k, we have

$$f(x_k) \geq f(\bar{x})$$

for such k. It follows that

$$f(x_k) - f(\bar{x}) = \frac{f^{(n)}(c_k)}{n!}(x_k-\bar{x})^n \geq 0.$$

This implies $f^{(n)}(c_k) \geq 0$ for such k. Since $\{c_k\}$ converges to $\bar{x}$, $f^{(n)}(\bar{x}) = \lim_{k\to\infty} f^{(n)}(c_k) \geq 0$.

The proof of (b) is similar. $\square$

■ **Example 4.5.2** Consider the function $f(x) = x^2 \cos x$ defined on $\mathbb{R}$. Then $f'(x) = 2x\cos x - x^2 \sin x$ and $f''(x) = 2\cos x - 4x\sin x - x^2\cos x$. Then $f(0) = f'(0) = 0$ and $f''(0) = 2 > 0$. It follows from the previous theorem that f has a local minimum at 0. Notice, by the way, that since $f(0) = 0$ and $f(\pi) < 0$, 0 is not a global minimum.

■ **Example 4.5.3** Consider the function $f(x) = -x^6 + 2x^5 + x^4 - 4x^3 + x^2 + 2x - 3$ defined on $\mathbb{R}$. A direct calculations shows $f'(1) = f''(1) = f'''(1) = f^{(4)}(1) = 0$ and $f^{(5)}(1) < 0$. It follows from the previous theorem that f has a local maximum at 1.

Exercises

4.5.1 ▷ Use Taylor's theorem to prove that

$$e^x > \sum_{k=0}^{m} \frac{x^k}{k!}$$

for all $x > 0$ and $m \in \mathbb{N}$.

4.5.2 Find the 5th Taylor polynomial, $P_5(x)$, at $\bar{x} = 0$ for $\cos x$. Determine an upper bound for the error $|P_5(x) - \cos x|$ for $x \in [-\pi/2, \pi/2]$.

4.5.3 Use Theorem 4.5.3 to determine if the following functions have a local minimum or a local maximum at the indicated points.

(a) $f(x) = x^3 \sin x$ at $\bar{x} = 0$.

(b) $f(x) = (1-x)\ln x$ at $\bar{x} = 1$.

4.5.4 Suppose f is twice differentiable on (a,b). Show that for every $x \in (a,b)$,

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

4.5.5 ▶ (a) Suppose f is three times differentiable on (a,b) and $\bar{x} \in (a,b)$. Prove that

$$\lim_{h\to 0} \frac{f(\bar{x}+h) - f(\bar{x}) - f'(\bar{x})\frac{h}{1!} - f''(\bar{x})\frac{h^2}{2!}}{h^3} = \frac{f'''(\bar{x})}{3!}.$$

(b) State and prove a more general result for the case where f is n times differentiable on (a,b).

4.5.6 Suppose f is n times differentiable on (a,b) and $\bar{x} \in (a,b)$. Define

$$P_n(h) = \sum_{k=0}^{n} f^{(n)}(\bar{x})\frac{h^n}{n!} \text{ for } h \in \mathbb{R}$$

Prove that

$$\lim_{h\to 0} \frac{f(\bar{x}+h) - P_n(h)}{h^n} = 0.$$

(Thus, we have

$$f(\bar{x}+h) = P_n(h) + g(h),$$

wher g is a function that satisfies $\lim_{h\to 0} \frac{g(h)}{h^n} = 0$. This is called the *Taylor expansion with Peano's remainder*.)

4.6 CONVEX FUNCTIONS AND DERIVATIVES

We discuss in this section a class of functions that plays an important role in optimization problems.

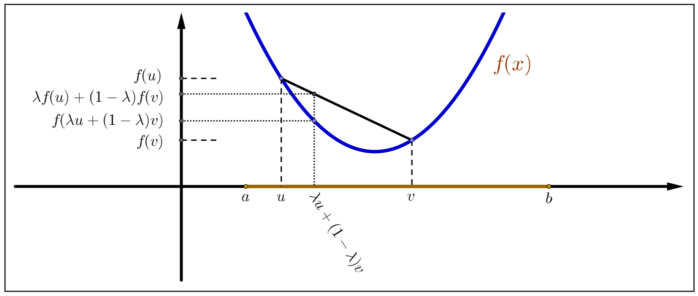

Figure 4.6: A Convex Function.

Definition 4.6.1 Let I be an interval of $\mathbb{R}$ and let $f\colon I \to \mathbb{R}$. We say that f is *convex* on I if

$$f(\lambda u + (1-\lambda)v) \le \lambda f(u) + (1-\lambda) f(v)$$

for all $u, v \in I$ and for all $\lambda \in (0,1)$.

■ **Example 4.6.1** The following functions are convex.

(a) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x$. This is straightforward.

(b) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Here note first that $2xy \le x^2 + y^2$ for all real numbers x, y. Then, if $0 < \lambda < 1$ and $x, y \in \mathbb{R}$, we get

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= (\lambda x + (1-\lambda)y)^2 \\ &= \lambda^2 x^2 + 2\lambda(1-\lambda)xy + (1-\lambda)^2 y^2 \\ &\le \lambda^2 x^2 + \lambda(1-\lambda)(x^2+y^2) + (1-\lambda)^2 y^2 \\ &= \lambda(\lambda x^2 + (1-\lambda)x^2) + (1-\lambda)(\lambda y^2 + (1-\lambda)y^2) \\ &= \lambda x^2 + (1-\lambda)y^2 \\ &= \lambda f(x) + (1-\lambda) f(y). \end{aligned}$$

(c) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$. This follows from the triangle inequality and other basic properties of absolute value.

Theorem 4.6.1 Let I be an interval of $\mathbb{R}$. A function $f\colon I \to \mathbb{R}$ is convex if and only if for every $\lambda_i \ge 0$, $i = 1, \ldots, n$, with $\sum_{i=1}^n \lambda_i = 1$ $(n \ge 2)$ and for every $x_i \in I$, $i = 1, \ldots, n$,

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \le \sum_{i=1}^n \lambda_i f(x_i). \tag{4.13}$$

Proof: Since the converse holds trivially, we only need to prove the implication by induction. The conclusion holds for $n = 2$ by the definition of convexity. Let k be such that the conclusion holds for any n with $2 \le n \le k$. We will show that it also holds for $n = k+1$. Fix $\lambda_i \ge 0$, $i = 1, \ldots, k+1$, with $\sum_{i=1}^{k+1} \lambda_i = 1$ and fix every $x_i \in I$, $i = 1, \ldots, k+1$. Then

$$\sum_{i=1}^k \lambda_i = 1 - \lambda_{k+1}.$$

If $\lambda_{k+1} = 1$, then $\lambda_i = 0$ for all $i = 1, \ldots, k$, and (4.13) holds. Suppose $0 \le \lambda_{k+1} < 1$. Then, for each $i = 1, \ldots, k$, $\lambda_i/(1-\lambda_{k+1}) \ge 0$ and

$$\sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} = 1.$$

It follows that

$$\begin{aligned} f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) &= f\left[(1-\lambda_{k+1})\frac{\sum_{i=1}^{k} \lambda_i x_i}{1-\lambda_{k+1}} + \lambda_{k+1} x_{k+1}\right] \\ &\le (1-\lambda_{k+1}) f\left(\frac{\sum_{i=1}^{k} \lambda_i x_i}{1-\lambda_{k+1}}\right) + \lambda_{k+1} f(x_{k+1}) \\ &= (1-\lambda_{k+1}) f\left(\sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} x_i\right) + \lambda_{k+1} f(x_{k+1}) \\ &\le (1-\lambda_{k+1}) \sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} f(x_i) + \lambda_{k+1} f(x_{k+1}) \\ &= \sum_{i=1}^{k+1} \lambda_i f(x_i), \end{aligned}$$

where the first inequality follows from the definition of convexity (or is trivial if $\lambda_{k+1} = 0$) and the last inequality follows from the inductive assumption. The proof is now complete. □

Theorem 4.6.2 Let I be an interval and let $f\colon I \to \mathbb{R}$ be a convex function. Then f has a local minimum at $\bar{x}$ if and only if f has an absolute minimum at $\bar{x}$.

Proof: Clearly if f has a global minimum at $\bar{x}$, then it also has a local minimum at $\bar{x}$.

Conversely, suppose that f has a local minimum at $\bar{x}$. Then there exists $\delta > 0$ such that

$$f(u) \ge f(\bar{x}) \text{ for all } u \in B(\bar{x};\delta) \cap I.$$

For any $x \in I$, we have $x_n = (1 - \frac{1}{n})\bar{x} + \frac{1}{n}x \to \bar{x}$. Thus, $x_n \in B(\bar{x};\delta) \cap I$ when n is sufficiently large. Thus, for such n,

$$f(\bar{x}) \le f(x_n) \le (1 - \frac{1}{n}) f(\bar{x}) + \frac{1}{n} f(x).$$

This implies that for a sufficient large n, we have

$$\frac{1}{n} f(\bar{x}) \le \frac{1}{n} f(x)$$

and, hence, $f(\bar{x}) \le f(x)$. Since x was arbitrary, this shows f has an absolute minimum at $\bar{x}$. □

Theorem 4.6.3 Let I be an open interval and let $f\colon I \to \mathbb{R}$ be a convex function. Suppose f is differentiable at $\bar{x}$. Then

$$f'(\bar{x})(x-\bar{x}) \le f(x) - f(\bar{x}) \text{ for all } x \in I. \tag{4.14}$$

Proof: For any $x \in I$ and $t \in (0,1)$, we have

$$\begin{aligned}\frac{f(\bar{x}+t(x-\bar{x}))-f(\bar{x})}{t} &= \frac{f(tx+(1-t)\bar{x})-f(\bar{x})}{t}\\ &\leq \frac{tf(x)+(1-t)f(\bar{x})-f(\bar{x})}{t}\\ &= f(x)-f(\bar{x}).\end{aligned}$$

Since f is differentiable at $\bar{x}$,

$$f'(\bar{x})(x-\bar{x}) = \lim_{t\to 0^+}\frac{f(\bar{x}+t(x-\bar{x}))-f(\bar{x})}{t} \leq f(x)-f(\bar{x}),$$

which completes the proof. □

Corollary 4.6.4 Let I be an open interval and let $f\colon I \to \mathbb{R}$ be a convex function. Suppose f is differentiable at $\bar{x}$. Then f has an absolute minimum at $\bar{x}$ if and only if $f'(\bar{x}) = 0$.
Proof: Suppose f has an absolute minimum at $\bar{x}$. By Theorem 4.2.1, $f'(\bar{x}) = 0$. Let us prove the converse. Suppose $f'(\bar{x}) = 0$. It follows from Theorem 4.6.3 that

$$0 = f'(\bar{x})(x-\bar{x}) \leq f(x)-f(\bar{x}) \text{ for all } x \in I.$$

This implies

$$f(x) \geq f(\bar{x}) \text{ for all } x \in I.$$

Thus, f has an absolute minimum at $\bar{x}$. □

Lemma 4.6.5 Let I be an open interval and suppose $f\colon I \to \mathbb{R}$ is a convex function. Fix $a,b,x \in I$ with $a < x < b$. Then

$$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(x)}{b-x}.$$

Proof: Let

$$t = \frac{x-a}{b-a}.$$

Then $t \in (0,1)$ and

$$f(x) = f(a+(x-a)) = f\left(a+\frac{x-a}{b-a}(b-a)\right) = f(a+t(b-a)) = f(tb+(1-t)a).$$

By convexity of f, we obtain

$$f(x) \leq tf(b)+(1-t)f(a).$$

Thus,

$$f(x)-f(a) \leq tf(b)+(1-t)f(a)-f(a) = t[f(b)-f(a)] = \frac{x-a}{b-a}(f(b)-f(a)).$$

Equivalently,

$$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a}.$$

Similarly,

$$f(x)-f(b) \leq tf(b)+(1-t)f(a)-f(b) = (1-t)[f(a)-f(b)] = \frac{x-b}{b-a}[f(b)-f(a)].$$

It follows that

$$\frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(x)}{b-x}.$$

The proof is now complete. □

Theorem 4.6.6 Let I be an open interval and let $f\colon I \to \mathbb{R}$ be a differentiable function. Then f is convex if and only if f' is increasing on I.

Proof: Suppose f is convex. Fix $a<b$ with $a,b \in I$. By Lemma 4.6.5, for any $x \in (a,b)$, we have

$$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a}.$$

This implies, taking limits, that

$$f'(a) \leq \frac{f(b)-f(a)}{b-a}.$$

Similarly,

$$\frac{f(b)-f(a)}{b-a} \leq f'(b).$$

Therefore, $f'(a) \leq f'(b)$, and f' is an increasing function.

Let us prove the converse. Suppose f' is increasing. Fix $x_1 < x_2$ and $t \in (0,1)$. Then

$$x_1 < x_t < x_2,$$

where $x_t = tx_1 + (1-t)x_2$. By the Mean Value Theorem (Theorem 4.2.3), there exist c_1 and c_2 such that

$$x_1 < c_1 < x_t < c_2 < x_2$$

with

$$f(x_t)-f(x_1) = f'(c_1)(x_t-x_1) = f'(c_1)(1-t)(x_2-x_1);$$
$$f(x_t)-f(x_2) = f'(c_2)(x_t-x_2) = f'(c_2)t(x_1-x_2).$$

This implies

$$tf(x_t)-tf(x_1) = f'(c_1)t(1-t)(x_2-x_1);$$
$$(1-t)f(x_t)-(1-t)f(x_2) = f'(c_2)t(1-t)(x_1-x_2).$$

Since $f'(c_1) \leq f'(c_2)$, we have

$$tf(x_t) - tf(x_1) = f'(c_1)t(1-t)(x_2 - x_1) \leq f'(c_2)t(1-t)(x_2 - x_1) = (1-t)f(x_2) - (1-t)f(x_t).$$

Rearranging terms, we get

$$f(x_t) \leq tf(x_1) + (1-t)f(x_2).$$

Therefore, f is convex. The proof is now complete. □

Corollary 4.6.7 Let I be an open interval and let $f\colon I \to \mathbb{R}$ be a function. Suppose f is twice differentiable on I. Then f is convex if and only if $f''(x) \geq 0$ for all $x \in I$.

Proof: It follows from Proposition 4.3.2 that $f''(x) \geq 0$ for all $x \in I$ if and only if the derivative function f' is increasing on I. The conclusion then follows directly from Theorem 4.6.6. □

■ **Example 4.6.2** Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sqrt{x^2+1}$. Now, $f'(x) = x/\sqrt{x^2+1}$ and $f''(x) = 1/(x^2+1)^{3/2}$. Since $f''(x) \geq 0$ for all x, it follows from the corollary that f is convex.

Theorem 4.6.8 Let I be an open interval and let $f\colon I \to \mathbb{R}$ be a convex function. Then it is locally Lipschitz continuous in the sense that for any $\bar{x} \in I$, there exist $\ell \geq 0$ and $\delta > 0$ such that

$$|f(u) - f(v)| \leq \ell|u - v| \text{ for all } u, v \in B(\bar{x}; \delta). \tag{4.15}$$

In particular, f is continuous.

Proof: Fix any $\bar{x} \in I$. Choose four numbers a, b, c, d satisfying

$$a < b < \bar{x} < c < d \text{ with } a, d \in I.$$

Choose $\delta > 0$ such that $B(\bar{x}; \delta) \subset (b, c)$. Let $u, v \in B(\bar{x}; \delta)$ with $v < u$. Then by Lemma 4.6.5, we see that

$$\frac{f(b)-f(a)}{b-a} \leq \frac{f(u)-f(a)}{u-a} \leq \frac{f(u)-f(v)}{u-v} \leq \frac{f(d)-f(v)}{d-v} \leq \frac{f(d)-f(c)}{d-c}.$$

Using a similar approach for the case $u < v$, we get

$$\frac{f(b)-f(a)}{b-a} \leq \frac{f(u)-f(v)}{u-v} \leq \frac{f(d)-f(c)}{d-c} \text{ for all } u, v \in B(\bar{x}; \delta).$$

Choose $\ell \geq 0$ sufficiently large so that

$$-\ell \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(u)-f(v)}{u-v} \leq \frac{f(d)-f(c)}{d-c} \leq \ell \text{ for all } u, v \in B(\bar{x}; \delta).$$

Then (4.15) holds. The proof is now complete. □

Exercises

4.6.1 (a) Let I be an interval and let $f,g\colon I \to \mathbb{R}$ be convex functions. Prove that cf, $f+g$, and $\max\{f,g\}$ are convex functions on I, where $c \geq 0$ is a constant.
(b) Find two convex functions f and g on an interval I such that $f \cdot g$ is not convex.

4.6.2 Let $f\colon \mathbb{R} \to \mathbb{R}$ be a convex function. Given $a,b \in \mathbb{R}$, prove that the function defined by

$$g(x) = f(ax+b), \text{ for } x \in \mathbb{R}$$

is also a convex function on $\mathbb{R}$.

4.6.3 ► Let I be an interval and let $f\colon I \to \mathbb{R}$ be a convex function. Suppose that ϕ is a convex, increasing function on an interval J that contains $f(I)$. Prove that $\phi \circ f$ is convex on I.

4.6.4 ▷ Prove that each of the following functions is convex on the given domain:

(a) $f(x) = e^{bx}, x \in \mathbb{R}$, where b is a constant.
(b) $f(x) = x^k$, $x \in [0,\infty)$ and $k \geq 1$ is a constant.
(c) $f(x) = -\ln(1-x)$, $x \in (-\infty, 1)$.
(d) $f(x) = -\ln\left(\dfrac{e^x}{1+e^x}\right)$, $x \in \mathbb{R}$.
(e) $f(x) = x\sin x$, $x \in (-\frac{\pi}{4}, \frac{\pi}{4})$.

4.6.5 ▷ Prove the following:

(a) If a,b are nonnegative real numbers, then

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

(b) If $a_1, a_2, \ldots, a_n$, where $n \geq 2$, are nonnegative real numbers, then

$$\frac{a_1+a_2+\cdots+a_n}{n} \geq (a_1 \cdot a_2 \cdots a_n)^{1/n}.$$

4.7 NONDIFFERENTIABLE CONVEX FUNCTIONS AND SUBDIFFERENTIALS

In this section, we introduce a new concept that is helpful in the study of optimization problems in which the objective function may fail to be differentiable.

Definition 4.7.1 Let $f\colon \mathbb{R} \to \mathbb{R}$ be a convex function. A number $u \in \mathbb{R}$ is called a *subderivative* of the function f at $\bar{x}$ if

$$u \cdot (x - \bar{x}) \leq f(x) - f(\bar{x}) \text{ for all } x \in \mathbb{R}. \tag{4.16}$$

The set of all subderivatives of f at $\bar{x}$ is called the *subdifferential* of f at $\bar{x}$ and is denoted by $\partial f(\bar{x})$.

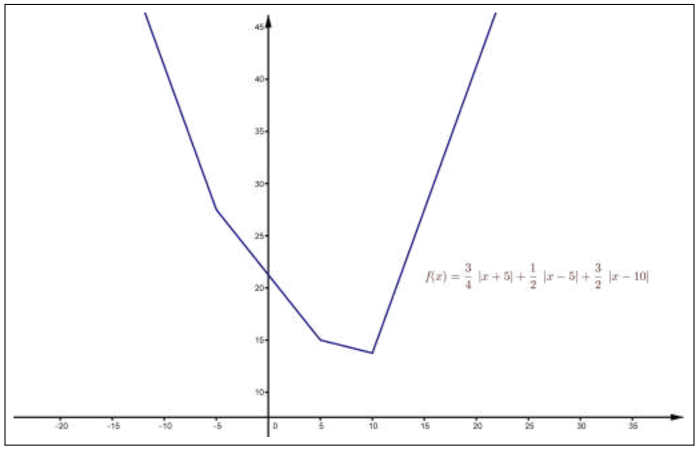

Figure 4.7: A nondifferential convex function.

■ **Example 4.7.1** Let $f(x) = |x|$. Then

$$\partial f(0) = [-1, 1].$$

Indeed, for any $u \in \partial f(0)$, we have

$$u \cdot x = u(x - 0) \le f(x) - f(0) = |x| \text{ for all } x \in \mathbb{R}.$$

In particular, $u \cdot 1 \le |1| = 1$ and $u \cdot (-1) = -u \le |-1| = 1$. Thus, $u \in [-1, 1]$. It follows that

$$\partial f(0) \subset [-1, 1].$$

For any $u \in [-1, 1]$, we have $|u| \le 1$. Then

$$u \cdot x \le |u \cdot x| = |u||x| \le |x| \text{ for all } x \in \mathbb{R}.$$

This implies $u \in \partial f(0)$. Therefore, $\partial f(0) = [-1, 1]$.

Lemma 4.7.1 Let $f\colon \mathbb{R} \to \mathbb{R}$ be a convex function. Fix $a \in \mathbb{R}$. Define the slope function ϕ_a by

$$\phi_a(x) = \frac{f(x) - f(a)}{x - a} \tag{4.17}$$

for $x \in (-\infty, a) \cup (a, \infty)$. Then, for $x_1, x_2 \in (-\infty, a) \cup (a, \infty)$ with $x_1 < x_2$, we have

$$\phi_a(x_1) \le \phi_a(x_2).$$

Proof: This lemma follows directly from Lemma 4.6.5. □

Theorem 4.7.2 Let $f\colon \mathbb{R} \to \mathbb{R}$ be a convex function and let $\bar{x} \in \mathbb{R}$. Then f has left derivative and right derivative at $\bar{x}$. Moreover,

$$\sup_{x < \bar{x}} \phi_{\bar{x}}(x) = f'_-(\bar{x}) \le f'_+(\bar{x}) = \inf_{x > \bar{x}} \phi_{\bar{x}}(x),$$

where $\phi_{\bar{x}}$ is defined in (4.17).

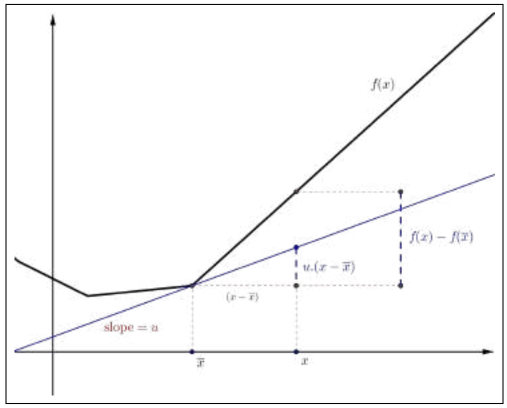

Figure 4.8: Definition of subderivative.

Proof: By Lemma 4.7.1, the slope function $\phi_{\bar{x}}$ defined by (4.17) is increasing on the interval $(\bar{x},\infty)$ and bounded below by $\phi_{\bar{x}}(\bar{x}-1)$. By Theorem 3.2.4, the limit

$$\lim_{x\to\bar{x}^+}\phi_{\bar{x}}(x) = \lim_{x\to\bar{x}^+}\frac{f(x)-f(\bar{x})}{x-\bar{x}}$$

exists and is finite. Moreover,

$$\lim_{x\to\bar{x}^+}\phi_{\bar{x}}(x) = \inf_{x>\bar{x}}\phi_{\bar{x}}(x).$$

Thus, $f'_+(\bar{x})$ exists and

$$f'_+(\bar{x}) = \inf_{x>\bar{x}}\phi_{\bar{x}}(x).$$

Similarly, $f'_-(\bar{x})$ exists and

$$f'_-(\bar{x}) = \sup_{x<\bar{x}}\phi_{\bar{x}}(x).$$

Applying Lemma 4.7.1 again, we see that

$$\phi_{\bar{x}}(x) \le \phi_{\bar{x}}(y) \text{ whenever } x<\bar{x}<y.$$

This implies $f'_-(\bar{x}) \le f'_+(\bar{x})$. The proof is complete. □

Theorem 4.7.3 Let $f\colon \mathbb{R}\to\mathbb{R}$ be a convex function and let $\bar{x}\in\mathbb{R}$. Then

$$\partial f(\bar{x}) = [f'_-(\bar{x}), f'_+(\bar{x})]. \tag{4.18}$$

Proof: Suppose $u\in\partial f(\bar{x})$. By the definition (4.16), we have

$$u\cdot(x-\bar{x}) \le f(x)-f(\bar{x}) \text{ for all } x>\bar{x}.$$

This implies

$$u \le \frac{f(x)-f(\bar{x})}{x-\bar{x}} \text{ for all } x>\bar{x}.$$

Thus,

$$u \leq \lim_{x \to \bar{x}^+} \frac{f(x) - f(\bar{x})}{x - \bar{x}} = f'_+(\bar{x}).$$

Similarly, we have

$$u \cdot (x - \bar{x}) \leq f(x) - f(\bar{x}) \text{ for all } x < \bar{x}.$$

Thus,

$$u \geq \frac{f(x) - f(\bar{x})}{x - \bar{x}} \text{ for all } x < \bar{x}.$$

This implies $u \geq f'_-(\bar{x})$. So

$$\partial f(\bar{x}) \subset [f'_-(\bar{x}), f'_+(\bar{x})].$$

To prove the opposite inclusion, take $u \in [f'_-(\bar{x}), f'_+(\bar{x})]$. By Theorem 4.7.2

$$\sup_{x<\bar{x}} \phi_{\bar{x}}(x) = f'_-(\bar{x}) \leq u \leq f'_+(\bar{x}) = \inf_{x>\bar{x}} \phi_{\bar{x}}(x).$$

Using the upper estimate by $f'_+(\bar{x})$ for u, one has

$$u \leq \phi_{\bar{x}}(x) = \frac{f(x) - f(\bar{x})}{x - \bar{x}} \text{ for all } x > \bar{x}.$$

It follows that

$$u \cdot (x - \bar{x}) \leq f(x) - f(\bar{x}) \text{ for all } x \geq \bar{x}.$$

Similarly, one also has

$$u \cdot (x - \bar{x}) \leq f(x) - f(\bar{x}) \text{ for all } x < \bar{x}.$$

Thus, (4.16) holds and, hence, $u \in \partial f(\bar{x})$. Therefore, (4.18) holds. □

Corollary 4.7.4 Let $f\colon \mathbb{R} \to \mathbb{R}$ be a convex function and $\bar{x} \in \mathbb{R}$. Then f is differentiable at $\bar{x}$ if and only if $\partial f(\bar{x})$ is a singleton. In this case,

$$\partial f(\bar{x}) = \{f'(\bar{x})\}.$$

Proof: Suppose f is differentiable at $\bar{x}$. Then

$$f'_-(\bar{x}) = f'_+(\bar{x}) = f'(\bar{x}).$$

By Theorem 4.7.3,

$$\partial f(\bar{x}) = [f'_-(\bar{x}), f'_+(\bar{x})] = \{f'(\bar{x})\}.$$

Thus, $\partial f(\bar{x})$ is a singleton.

Conversely, if $\partial f(\bar{x})$ is a singleton, we must have $f'_-(\bar{x}) = f'_+(\bar{x})$. Thus, f is differentiable at $\bar{x}$.
□

■ **Example 4.7.2** Let $f(x) = a|x-b|+c$, where $a>0$. Then f is a convex function and

$$f'_-(b) = -a,\ f'_+(b) = a.$$

Thus,

$$\partial f(b) = [-a,a].$$

Since f is differentiable on $(-\infty,b)$ and (b,∞), we have

$$\partial f(x) = \begin{cases} \{-a\}, & \text{if } x<b; \\ [-a,a], & \text{if } x=b; \\ \{a\}, & \text{if } x>b. \end{cases}$$

Definition 4.7.2 Let A and B be two nonempty subsets of $\mathbb{R}$ and let $\alpha\in\mathbb{R}$. Define

$$A+B = \{a+b : a\in A, b\in B\} \text{ and } \alpha A = \{\alpha a : a\in A\}.$$

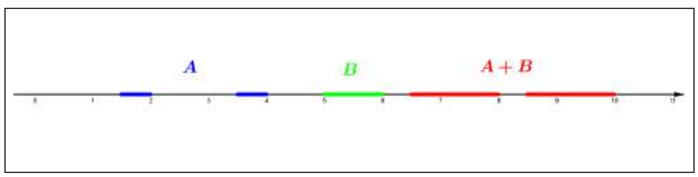

Figure 4.9: Set addition.

Theorem 4.7.5 Let $f,g\colon \mathbb{R}\to\mathbb{R}$ be convex functions and let $\alpha>0$. Then $f+g$ and αf are convex functions and

$$\partial(f+g)(\bar{x}) = \partial f(\bar{x}) + \partial g(\bar{x})$$
$$\partial(\alpha f)(\bar{x}) = \alpha\partial f(\bar{x}).$$

Proof: It is not hard to see that $f+g$ is a convex function and

$$(f+g)'_+(\bar{x}) = f'_+(\bar{x}) + g'_+(\bar{x})$$
$$(f+g)'_-(\bar{x}) = f'_-(\bar{x}) + g'_-(\bar{x}).$$

By Theorem 4.7.3,

$$\begin{aligned}\partial(f+g)(\bar{x}) &= [(f+g)'_-(\bar{x}), (f+g)'_+(\bar{x})] \\ &= [f'_-(\bar{x}) + g'_-(\bar{x}), f'_+(\bar{x}) + g'_+(\bar{x})] \\ &= [f'_-(\bar{x}), f'_+(\bar{x})] + [g'_-(\bar{x}), g'_+(\bar{x})] \\ &= \partial f(\bar{x}) + \partial g(\bar{x}).\end{aligned}$$

The proof for the second formula is similar. □

■ **Example 4.7.3** Let $a_1 < a_2 < \cdots < a_n$ and let $\mu_i > 0$ for $i = 1, \ldots, n$. Define

$$f(x) = \sum_{i=1}^{n} \mu_i |x - a_i|.$$

Then f is a convex function. By Theorem 4.7.5, we get

$$\partial f(\bar{x}) = \begin{cases} \sum_{a_i<\bar{x}} \mu_i - \sum_{a_i>\bar{x}} \mu_i, & \text{if } \bar{x} \notin \{a_1, \ldots, a_n\} \\ \sum_{a_i<\bar{x}} \mu_i - \sum_{a_i>\bar{x}} \mu_i + [-\mu_{i_0}, \mu_{i_0}], & \text{if } \bar{x} = a_{i_0}. \end{cases}$$

Theorem 4.7.6 Let $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$, be convex functions. Define

$$f(x) = \max\{f_i(x) : i = 1, \ldots, n\} \text{ and } I(u) = \{i = 1, \ldots, n : f_i(u) = f(u)\}.$$

Then f is a convex function. Moreover,

$$\partial f(\bar{x}) = [m, M],$$

where

$$m = \min_{i \in I(\bar{x})} f'_{i-}(\bar{x}) \text{ and } M = \max_{i \in I(\bar{x})} f'_{i+}(\bar{x}).$$

Proof: Fix $u, v \in \mathbb{R}$ and $\lambda \in (0, 1)$. For any $i = 1, \ldots, n$, we have

$$f_i(\lambda u + (1-\lambda)v) \leq \lambda f_i(u) + (1-\lambda) f_i(v) \leq \lambda f(u) + (1-\lambda) f(v).$$

This implies

$$f(\lambda u + (1-\lambda)v) = \max_{1 \leq i \leq n} f_i(\lambda u + (1-\lambda)v) \leq \lambda f(u) + (1-\lambda) f(v).$$

Thus, f is a convex function. Similarly we verify that $f'_+(\bar{x}) = M$ and $f'_-(\bar{x}) = m$. By Theorem 4.7.3,

$$\partial f(\bar{x}) = [m, M].$$

The proof is now complete. □

Remark 4.7.7 The product of two convex functions is not a convex function in general. For instance, $f(x) = x$ and $g(x) = x^2$ are convex functions, but $h(x) = x^3$ is not a convex function.

The following result may be considered as a version of the first derivative test for extrema in the case of non differentiable functions.

Theorem 4.7.8 Let $f\colon \mathbb{R} \to \mathbb{R}$ be a convex function. Then f has an absolute minimum at $\bar{x}$ if and only if

$$0 \in \partial f(\bar{x}) = [f'_-(\bar{x}), f'_+(\bar{x})].$$

Proof: Suppose f has an absolute minimum at $\bar{x}$. Then

$$f(\bar{x}) \leq f(x) \text{ for all } x \in \mathbb{R}.$$

This implies

$$0 \cdot (x - \bar{x}) = 0 \leq f(x) - f(\bar{x}) \text{ for all } x \in \mathbb{R}.$$

It follows from (4.16) that $0 \in \partial f(\bar{x})$.

Conversely, if $0 \in \partial f(\bar{x})$, again, by (4.16),

$$0 \cdot (x - \bar{x}) = 0 \leq f(x) - f(\bar{x}) \text{ for all } x \in \mathbb{R}.$$

Thus, f has an absolute minimum at $\bar{x}$. □

■ **Example 4.7.4** Let k be a positive integer and $a_1 < a_2 < \cdots < a_{2k-1}$. Define

$$f(x) = \sum_{i=1}^{2k-1} |x - a_i|,$$

for $x \in \mathbb{R}$. It follows from the subdifferential formula in Example 4.7.3 that $0 \in \partial f(\bar{x})$ if and only if $\bar{x} = a_k$. Thus, f has a unique absolute minimum at a_k.

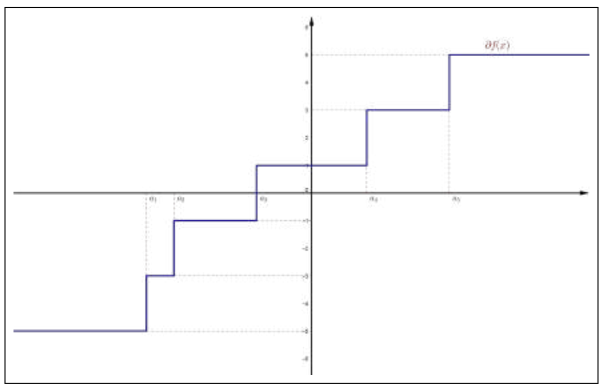

Figure 4.10: Subdifferential of $f(x) = \sum_{i=1}^{2k-1} |x - a_i|$.

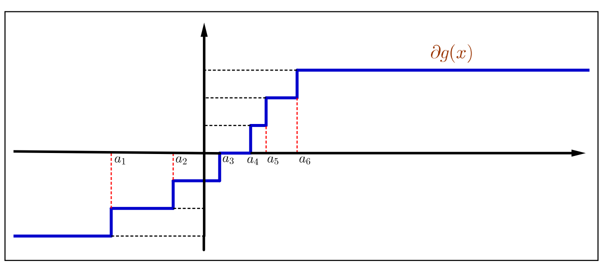

Figure 4.11: Subdifferential of $g(x) = \sum_{i=1}^{2k} |x - a_i|$.

Similarly, if $a_1 < a_2 < \cdots < a_{2k}$ and

$$g(x) = \sum_{i=1}^{2k} |x - a_i|.$$

Then $0 \in \partial g(\bar{x})$ if and only if $\bar{x} \in [a_k, a_{k+1}]$. Thus, g has an absolute minimum at any point of $[a_k, a_{k+1}]$.

The following theorem is a version of the Mean Value Theorem (Theorem 4.2.3) for nondifferentiable functions.

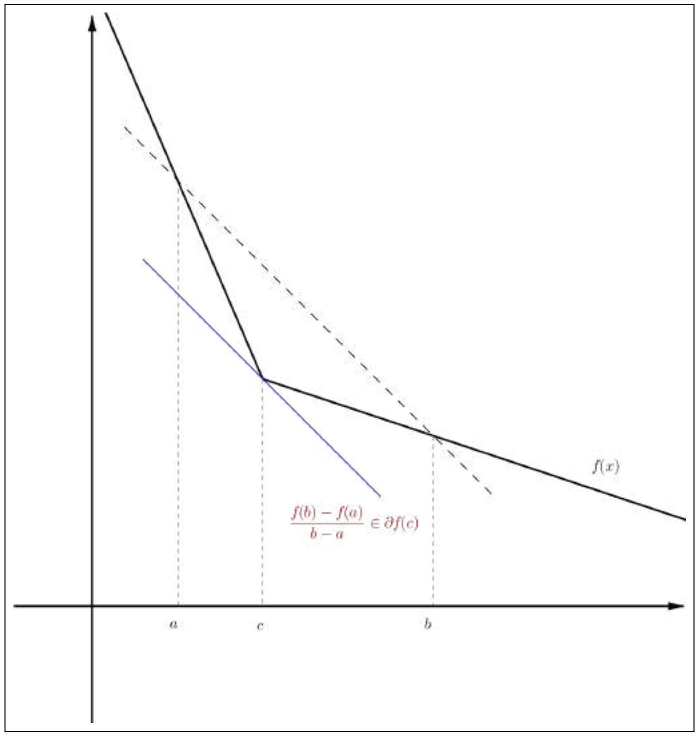

Figure 4.12: Subdifferential mean value theorem.

Theorem 4.7.9 Let $f\colon \mathbb{R} \to \mathbb{R}$ be a convex function and let $a < b$. Then there exists $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} \in \partial f(c). \tag{4.19}$$

Proof: Define

$$g(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right].$$

Then g is a convex function and $g(a) = g(b)$. Thus, g has a local minimum at some $c \in (a,b)$ and, hence, g also has an absolute minimum at c. Observe that the function

$$h(x) = -\left[\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right]$$

is differentiable at c and, hence,

$$\partial h(c) = \{h'(c)\} = \left\{-\frac{f(b)-f(a)}{b-a}\right\}.$$

By Theorem 4.7.8 and the subdifferential sum rule,

$$0 \in \partial g(c) = \partial f(c) - \left\{ \frac{f(b)-f(a)}{b-a} \right\}.$$

This implies (4.19). The proof is now complete. □

Corollary 4.7.10 Let $f\colon \mathbb{R} \to \mathbb{R}$ be a convex function. Then f is Lipschitz continuous if and only if there exists $\ell \geq 0$ such that

$$\partial f(x) \subset [-\ell, \ell] \text{ for all } x \in \mathbb{R}.$$

Proof: Suppose f is Lipschitz continuous on $\mathbb{R}$. Then there exists $\ell \geq 0$ such that

$$|f(u) - f(v)| \leq \ell |u - v| \text{ for all } u, v \in \mathbb{R}.$$

Then for any $x \in \mathbb{R}$,

$$f'_+(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \leq \lim_{h \to 0^+} \frac{\ell |h|}{h} = \ell.$$

Similarly, $f'_-(x) \geq -\ell$. Thus,

$$\partial f(x) = [f'_-(x), f'_+(x)] \subset [-\ell, \ell].$$

Conversely, fix any $u, v \in \mathbb{R}$ with $u \neq v$. Applying Theorem 4.7.9, we get

$$\frac{f(v) - f(u)}{v - u} \in \partial f(c) \subset [-\ell, \ell],$$

for some c in between u and v. This implies

$$|f(u) - f(v)| \leq \ell |u - v|.$$

This inequality obviously holds for $u = v$. Therefore, f is Lipschitz continuous. □

Exercises

4.7.1 ▷ Find subdifferentials of the following functions:

(a) $f(x) = a|x|$, $a > 0$.
(b) $f(x) = |x - 1| + |x + 1|$.

4.7.2 Find the subdifferential of the function

$$f(x) = \max\{-2x + 1, x, 2x - 1\}.$$

4.7.3 ▶ Let $f(x) = \sum_{k=1}^{n} |x - k|$. Find all absolute minimizers of the function.

4.7.4 Let $f\colon\ \mathbb{R} \to \mathbb{R}$ be a convex function. Fix $a, b \in \mathbb{R}$ and define the function g by

$$g(x) = f(ax+b),\ \text{for } x \in \mathbb{R}$$

Prove that $\partial g(\bar{x}) = a\partial f(a\bar{x}+b)$.

4.7.5 ▷ Let $f\colon\ \mathbb{R} \to \mathbb{R}$ be a convex function. Suppose that $\partial f(x) \subset [0,\infty)$ for all $x \in \mathbb{R}$. Prove that f is monotone increasing on $\mathbb{R}$.

5. Solutions and Hints for Selected Exercises

SECTION 1.1

Exercise 1.1.2. Applying basic rules of operations on sets yields

$$(X \setminus Y) \cap Z = Y^c \cap Z = Z \setminus Y.$$

and

$$Z \setminus (Y \cap Z) = (Z \setminus Y) \cup (Z \setminus Z) = (Z \setminus Y) \cup \emptyset = Z \setminus Y.$$

Therefore, $(X \setminus Y) \cap Z = Z \setminus (Y \cap Z)$.

SECTION 1.2

Exercise 1.2.1. (a) For any $a \in A$, we have $f(a) \in f(A)$ and, so, $a \in f^{-1}(f(A))$. This implies $A \subset f^{-1}(f(A))$. Note that this inclusion does not require the injectivity of f. Now fix any $a \in f^{-1}(f(A))$. Then $f(a) \in f(A)$, so there exists $a' \in A$ such that $f(a) = f(a')$. Since f is one-to-one, $a = a' \in A$. Therefore, $f^{-1}(f(A)) \subset A$ and the equality holds.

(b) Fix any $b \in f(f^{-1}(B))$. Then $b = f(x)$ for some $x \in f^{-1}(B)$. Thus, $b = f(x) \in B$ and, hence, $f(f^{-1}(B)) \subset B$. This inclusion does not require the surjectivity of f. Now fix $b \in B$. Since f is onto, there exists $x \in X$ such that $f(x) = b \in B$. Thus, $x \in f^{-1}(B)$ and, hence, $b \in f(f^{-1}(B))$. We have shown that $B \subset f(f^{-1}(B))$ and the equality holds.

Without the injectivity of f, the equality in part (a) is no longer valid. Consider $f(x) = x^2$, $x \in \mathbb{R}$, and $A = [-1, 2]$. Then $f(A) = [0, 4]$ and, hence, $f^{-1}(f(A)) = [-2, 2]$, which strictly contains A. It is also not hard to find an example of a function f and a set B for which the equality in part (b) does not hold true.

SECTION 1.3

Exercise 1.3.6. For $n = 1$,

$$\frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right] = \frac{1}{\sqrt{5}}\frac{2\sqrt{5}}{2} = 1.$$

Thus, the conclusion holds for $n = 1$. It is also easy to verify that the conclusion holds for $n = 2$.

Suppose that

$$a_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$$

for all $k \le n$, where $n \ge 2$. Let us show that

$$a_{n+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]. \tag{5.1}$$

By the definition of the sequence and the induction hypothesis,

$$\begin{aligned}
a_{n+1} &= a_n + a_{n-1} \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{1+\sqrt{5}}{2}+1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{1-\sqrt{5}}{2}-1\right)\right].
\end{aligned}$$

Observe that

$$\frac{1+\sqrt{5}}{2}+1 = \frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2 \text{ and } \frac{1-\sqrt{5}}{2}+1 = \frac{3-\sqrt{5}}{2} = \left(\frac{1-\sqrt{5}}{2}\right)^2.$$

Therefore, (5.1) follows easily.

In this exercise, observe that the two numbers $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$ are the roots of the quadratic equation

$$x^2 = x + 1.$$

A more general result can be formulated as follows. Consider the sequence $\{a_n\}$ defined by

$$\begin{aligned}
a_1 &= a; \\
a_2 &= b; \\
a_{n+2} &= \alpha a_{n+1} + \beta a_n \text{ for } n \in \mathbb{N}.
\end{aligned}$$

Suppose that the equation $x^2 = \alpha x + \beta$ has two solutions x_1 and x_2. Let c_1 and c_2 be two constants such that

$$\begin{aligned}
c_1 x_1 + c_2 x_2 &= a; \\
c_1(x_1)^2 + c_2(x_2)^2 &= b.
\end{aligned}$$

Then we can prove by induction that

$$x_n = c_1(x_1)^n + c_2(x_2)^n \text{ for all } n \in \mathbb{N}.$$

This is a very useful method to find a general formula for a sequence defined recursively as above. For example, consider the sequence

$$a_1 = 1;$$
$$a_2 = 1;$$
$$a_{n+2} = a_{n+1} + 2a_n \text{ for } n \in \mathbb{N}.$$

Solving the equation $x^2 = x + 2$ yields two solutions $x_1 = 2$ and $x_2 = (-1)$. Thus,

$$x_n = c_1 2^n + c_2(-1)^n,$$

where c_1 and c_2 are constants such as

$$c_1(2) + c_2(-1) = 1;$$
$$c_1(2)^2 + c_2(-1)^2 = 1.$$

It is not hard to see that $c_1 = 1/3$ and $c_2 = -1/3$. Therefore,

$$a_n = \frac{1}{3}2^n - \frac{1}{3}(-1)^n \text{ for all } n \in \mathbb{N}.$$

Exercise 1.3.8. *Hint:* Prove first that, for $k = 1, 2, \ldots, n$, we have

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

SECTION 1.4

Exercise 1.4.5. In general, to prove that $|a| \le m$, where $m \ge 0$, we only need to show that $a \le m$ and $-a \le m$.

For any $x, y \in \mathbb{R}$,

$$|x| = |x - y + y| \le |x - y| + |y|,$$

This implies

$$|x| - |y| \le |x - y|.$$

Similarly,

$$|y| = |y - x + x| \le |x - y| + |x|,$$

This implies

$$-(|x| - |y|) \le |x - y|.$$

Therefore,

$$||x| - |y|| \le |x - y|.$$

SECTION 1.5

Exercise 1.5.4.

Let us first show that $A+B$ is bounded above. Since A and B are nonempty and bounded above, by the completeness axiom, $\sup A$ and $\sup B$ exist and are real numbers. In particular, $a \leq \sup A$ for all $a \in A$ and $b \leq \sup B$ for all $b \in B$.

For any $x \in A+B$, there exist $a \in A$ and $b \in B$ such that $x = a+b$. Thus, $x = a+b \leq \sup A + \sup B$, which shows that $A+B$ is bounded above.

We will now show that $\sup A + \sup B$ is the supremum of the set $A+B$ by showing that $\sup A + \sup B$ satisfies conditions (1') and (2') of Proposition 1.5.1.

We have just shown that $\sup A + \sup B$ is an upper bound of $A+B$ and, hence, $\sup A + \sup B$ satisfies condition (1').

Now let $\varepsilon > 0$. Using $\frac{\varepsilon}{2}$ in part (2') of Proposition 1.5.1 applied to the sets A and B, there exits $a \in A$ and $b \in B$ such that

$$\sup A - \frac{\varepsilon}{2} < a \text{ and } \sup B - \frac{\varepsilon}{2} < b.$$

It follows that

$$\sup A + \sup B - \varepsilon < a+b.$$

This proves condition (2'). It follows from Proposition 1.5.1 applied to the set $A+B$ that $\sup A + \sup B = \sup(A+B)$ as desired.

SECTION 1.6

Exercise 1.6.2. Let $x = \dfrac{1}{r}$. By Theorem 1.6.2(d), there exists $m \in \mathbb{Z}$ such that

$$m-1 \leq \frac{1}{r} < m.$$

Since $1/r > 1$, we get $m > 1$ and, so, $m \geq 2$. It follows that $m-1 \in \mathbb{N}$. Set $n = m-1$ and then we get

$$\frac{1}{n+1} < r \leq \frac{1}{n}.$$

SECTION 2.1

Exercise 2.1.12. (a) Suppose that $\lim_{n\to\infty} a_n = \ell$. Then by Theorem 2.1.9,

$$\lim_{n\to\infty} a_{2n} = \ell \text{ and } \lim_{n\to\infty} a_{2n+1} = \ell. \tag{5.2}$$

Now suppose that (5.2) is satisfied. Fix any $\varepsilon > 0$. Choose $N_1 \in \mathbb{N}$ such that

$$|a_{2n} - \ell| < \varepsilon \text{ whenever } n \geq N_1,$$

and choose $N_2 \in \mathbb{N}$ such that

$$|a_{2n+1} - \ell| < \varepsilon \text{ whenever } n \geq N_2.$$

Let $N = \max\{2N_1, 2N_2+1\}$. Then

$$|a_n - \ell| < \varepsilon \text{ whenever } n \geq N.$$

Therefore, $\lim_{n\to\infty} a_n = \ell$.

This problem is sometimes very helpful to show that a limit exists. For example, consider the sequence defined by

$$x_1 = 1/2,$$
$$x_{n+1} = \frac{1}{2+x_n} \text{ for } n \in \mathbb{N}.$$

We will see later that $\{x_{2n+1}\}$ and $\{x_{2n}\}$ both converge to $\sqrt{2}-1$, so we can conclude that $\{x_n\}$ converges to $\sqrt{2}-1$.

(b) Use a similar method to the solution of part (a).

Exercise 2.1.8. Consider the case where $\ell > 0$. By the definition of limit, we can find $n_1 \in \mathbb{N}$ such that

$$|a_n| > \ell/2 \text{ for all } n \geq n_1.$$

Given any $\varepsilon > 0$, we can find $n_2 \in \mathbb{N}$ such that

$$|a_n - \ell| < \frac{\ell\varepsilon}{4} \text{ for all } n \geq n_2.$$

Choose $n_0 = \max\{n_1, n_2\}$. For any $n \geq n_0$, one has

$$\left|\frac{a_{n+1}}{a_n} - 1\right| = \frac{|a_n - a_{n+1}|}{|a_n|} \leq \frac{|a_n - \ell| + |a_{n+1} - \ell|}{|a_n|} < \frac{\frac{\ell\varepsilon}{4} + \frac{\ell\varepsilon}{4}}{\frac{\ell}{2}} = \varepsilon.$$

Therefore, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. If $\ell < 0$, consider the sequence $\{-a_n\}$.

The conclusion is no longer true if $\ell = 0$. A counterexample is $a_n = \lambda^n$ where $\lambda \in (0,1)$.

SECTION 2.2

Exercise 2.2.3. (a) The limit is calculated as follows:

$$\begin{aligned}
\lim_{n\to\infty}\left(\sqrt{n^2+n} - n\right) &= \lim_{n\to\infty} \frac{\left(\sqrt{n^2+n} - n\right)\left(\sqrt{n^2+n} + n\right)}{\sqrt{n^2+n}+n} \\
&= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} \\
&= \lim_{n\to\infty} \frac{n}{\sqrt{n^2(1+1/n)}+n} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n}+1} = 1/2.
\end{aligned}$$

(b) The limit is calculated as follows:

$$\begin{aligned}
\lim_{n\to\infty}\left(\sqrt[3]{n^3+3n^2}-n\right) &= \lim_{n\to\infty}\frac{\left(\sqrt[3]{n^3+3n^2}-n\right)\left(\sqrt[3]{(n^3+3n^2)^2}+n\sqrt[3]{n^3+3n^2}+n^2\right)}{\sqrt[3]{(n^3+3n^2)^2}+n\sqrt[3]{n^3+3n^2}+n^2)}\\
&= \lim_{n\to\infty}\frac{3n^2}{\sqrt[3]{(n^3+3n^2)^2}+n\sqrt[3]{n^3+3n^2}+n^2}\\
&= \lim_{n\to\infty}\frac{3n^2}{\sqrt[3]{n^6(1+3/n)^2}+n\sqrt[3]{n^3(1+3/n)}+n^2}\\
&= \lim_{n\to\infty}\frac{3n^2}{n^2\left(\sqrt[3]{(1+3/n)^2}+\sqrt[3]{(1+3/n)}+1\right)}\\
&= \lim_{n\to\infty}\frac{3}{\left(\sqrt[3]{(1+3/n)^2}+\sqrt[3]{(1+3/n)}+1\right)}=1.
\end{aligned}$$

(c) We use the result in par (a) and part (b) to obtain

$$\begin{aligned}
\lim_{n\to\infty}(\sqrt[3]{n^3+3n^2}-\sqrt{n^2+1}) &= \lim_{n\to\infty}\left(\sqrt[3]{n^3+3n^2}-n+n-\sqrt{n^2+1}\right)\\
&= \lim_{n\to\infty}\left(\sqrt[3]{n^3+3n^2}-n\right)+\lim_{n\to\infty}\left(n-\sqrt{n^2+1}\right)=1-1/2=1/2.
\end{aligned}$$

Using a similar technique, we can find the following limit:

$$\lim_{n\to\infty}\left(\sqrt[3]{an^3+bn^2+cn+d}-\sqrt{\alpha n^2+\beta n+\gamma}\right),$$

where $a>0$ and $\alpha>0$.

SECTION 2.3

Exercise 2.3.1. (a) Clearly, $a_1<2$. Suppose that $a_k<2$ for $k\in\mathbb{N}$. Then

$$a_{k+1}=\sqrt{2+a_k}<\sqrt{2+2}=2.$$

By induction, $a_n<2$ for all $n\in\mathbb{N}$.

(b) Clearly, $a_1=\sqrt{2}<\sqrt{2+\sqrt{2}}=a_2$. Suppose that $a_k<a_{k+1}$ for $k\in\mathbb{N}$. Then

$$a_k+2<a_{k+1}+2,$$

which implies

$$\sqrt{a_k+2}<\sqrt{a_{k+1}+2}.$$

Thus, $a_{k+1}<a_{k+2}$. By induction, $a_n<a_{n+1}$ for all $n\in\mathbb{N}$. Therefore, $\{a_n\}$ is an increasing sequence.

(c) By the monotone convergence theorem, $\lim_{n\to\infty}a_n$ exists. Let $\ell=\lim_{n\to\infty}a_n$. Since $a_{n+1}=\sqrt{2+a_n}$ and $\lim_{n\to\infty}a_{n+1}=\ell$, we have

$$\ell=\sqrt{2+\ell}\text{ or }\ell^2=2+\ell.$$

Solving this quadratic equation yields $\ell = -1$ or $\ell = 2$. Therefore, $\lim_{n\to\infty} a_n = 2$.

Define a more general sequence as follows:

$$a_1 = c > 0,$$
$$a_{n+1} = \sqrt{c + a_n} \quad \text{for } n \in \mathbb{N}.$$

We can prove that $\{a_n\}$ is monotone increasing and bounded above by $\dfrac{1+\sqrt{1+4c}}{2}$. In fact, $\{a_n\}$ converges to this limit. The number $\dfrac{1+\sqrt{1+4c}}{2}$ is obtained by solving the equation $\ell = \sqrt{c+\ell}$, where $\ell > 0$.

Exercise 2.3.2. (a) The limit is 3.
(b) The limit is 3.
(c) We use the well-known inequality

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc} \text{ for } a, b, c \geq 0.$$

By induction, we see that $a_n > 0$ for all $n \in \mathbb{N}$. Moreover,

$$a_{n+1} = \frac{1}{3}(2a_n + \frac{1}{a_n^2}) = \frac{1}{3}(a_n + a_n + \frac{1}{a_n^2}) \geq \frac{1}{3}\sqrt[3]{a_n \cdot a_n \cdot \frac{1}{a_n^2}} = 1.$$

We also have, for $n \geq 2$,

$$a_{n+1} - a_n = \frac{1}{3}\left(2a_n + \frac{1}{a_n^2}\right) - a_n = \frac{-a_n^3 + 1}{3a_n^2} = \frac{-(a_n - 1)(a_n^2 + a_n + 1)}{3a_n^2} < 0.$$

Thus, $\{a_n\}$ is monotone deceasing (for $n \geq 2$) and bounded below. We can show that $\lim_{n\to\infty} a_n = 1$.
(d) Use the inequality $\frac{a+b}{2} \geq \sqrt{ab}$ for $a, b \geq 0$ to show that $a_{n+1} \geq \sqrt{b}$ for all $n \in \mathbb{N}$. Then follow part 3 to show that $\{a_n\}$ is monotone decreasing. The limit is $\sqrt{b}$.

Exercise 2.3.3. (a) Let $\{a_n\}$ be the given sequence. Observe that $a_{n+1} = \sqrt{2a_n}$. Then show that $\{a_n\}$ is monotone increasing and bounded above. The limit is 2.
(b) Let $\{a_n\}$ be the given sequence. Then

$$a_{n+1} = \frac{1}{2 + a_n}.$$

Show that $\{a_{2n+1}\}$ is monotone decreasing and bounded below; $\{a_{2n}\}$ is monotone increasing and bounded above. Thus, $\{a_n\}$ converges by Exercise 2.1.12. The limit is $\sqrt{2} - 1$.

Exercise 2.3.5. Observe that

$$b_{n+1} = \frac{a_n + b_n}{2} \geq \sqrt{a_n b_n} = a_{n+1} \text{ for all } n \in \mathbb{N}.$$

Thus,

$$a_{n+1} = \sqrt{a_n b_n} \geq \sqrt{a_n a_n} = a_n \text{ for all } n \in \mathbb{N},$$

$$b_{n+1} = \frac{a_n + b_n}{2} \le \frac{b_n + b_n}{2} = b_n \text{ for all } n \in \mathbb{N}.$$

It follows that $\{a_n\}$ is monotone increasing and bounded above by b_1, and $\{b_n\}$ is decreasing and bounded below by a_1. Let $x = \lim_{n\to\infty} a_n$ and $y = \lim_{n\to\infty} b_n$. Then

$$x = \sqrt{xy} \text{ and } y = \frac{x+y}{2}.$$

Therefore, $x = y$.

SECTION 2.4

Exercise 2.4.1. Here we use the fact that in $\mathbb{R}$ a sequence is a Cauchy sequence if and only if it is convergent.

(a) Not a Cauchy sequence. See Example 2.1.7.

(b) A Cauchy sequence. This sequence converges to 0.

(c) A Cauchy sequence. This sequence converges to 1.

(d) A Cauchy sequence. This sequence converges to 0 (see Exercise 2.1.5).

SECTION 2.5

Exercise 2.5.4. (a) Define

$$\alpha_n = \sup_{k\ge n}(a_n + b_n),\ \beta_n = \sup_{k\ge n} a_k,\ \gamma_n = \sup_{k\ge n} b_k.$$

By the definition,

$$\limsup_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty}\alpha_n,\ \limsup_{n\to\infty} a_n = \lim_{n\to\infty}\beta_n,\ \limsup_{n\to\infty} b_n = \lim_{n\to\infty}\gamma_n.$$

By Exercise 2.5.3,

$$\alpha_n \le \beta_n + \gamma_n \text{ for all } n \in \mathbb{N}.$$

This implies

$$\lim_{n\to\infty}\alpha_n \le \lim_{n\to\infty}\beta_n + \lim_{n\to\infty}\gamma_n \text{ for all } n \in \mathbb{N}.$$

Therefore,

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

This conclusion remains valid for unbounded sequences provided that the right-hand side is well-defined. Note that the right-hand side is not well-defined, for example, when $\limsup_{n\to\infty} a_n = \infty$ and $\limsup_{n\to\infty} b_n = -\infty$.

(b) Define

$$\alpha_n = \inf_{k\ge n}(a_n + b_n),\ \beta_n = \inf_{k\ge n} a_k,\ \gamma_n = \inf_{k\ge n} b_k.$$

Proceed as in part (a), but use part (b) of Exercise 2.5.3.

(c) Consider $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$.

SECTION 2.6

Exercise 2.6.3. Suppose A and B are compact subsets of $\mathbb{R}$. Then, by Theorem 2.6.5, A and B are closed and bounded. From Theorem 2.6.2(c) we get that $A \cup B$ is closed. Moreover, let M_A, m_A, M_B, m_B be upper and lower bounds for A and B, respectively. Then $M = \max\{M_A, M_B\}$ and $m = \min\{m_A, m_B\}$ are upper and lower bounds for $A \cup B$. In particular, $A \cup B$ is bounded. We have shown that $A \cup B$ is both closed and bounded. It now follows from Theorem 2.6.5 that $A \cup B$ is compact.

SECTION 3.1

Exercise 3.1.6. (a) Observe that when x is near $1/2$, $f(x)$ is near $1/2$ no matter whether x is rational or irrational. We have

$$|f(x) - 1/2| = \begin{cases} |x - 1/2|, & \text{if } x \in \mathbb{Q}; \\ |1 - x - 1/2|, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Thus, $|f(x) - 1/2| = |x - 1/2|$ for all $x \in \mathbb{R}$.

Given any $\varepsilon > 0$, choose $\delta = \varepsilon$. Then

$$|f(x) - 1/2| < \varepsilon \text{ whenever } |x - 1/2| < \delta.$$

Therefore, $\lim_{x \to 1/2} f(x) = 1/2$.

(b) Observe that when x is near 0 and x is rational, $f(x)$ is near 0. However, when f is near 0 and x is irrational, $f(x)$ is near 1. Thus, the given limit does not exists. We justify this using the sequential criterion for limits (Theorem 3.1.2). By contradiction, assume that

$$\lim_{x \to 0} f(x) = \ell,$$

where ℓ is a real number. Choose a sequence $\{r_n\}$ of rational numbers that converges to 0, and choose also a sequence $\{s_n\}$ of irrational numbers that converges to 0. Then $f(r_n) = r_n$ and $f(s_n) = 1 - s_n$ and, hence,

$$\ell = \lim_{n \to \infty} f(r_n) = 0$$

and

$$\ell = \lim_{n \to \infty} f(s_n) = \lim_{n \to \infty} (1 - s_n) = 1.$$

This is a contradiction.

(c) By a similar method to part (b), we can show that $\lim_{x \to 1} f(x)$ does not exists.

Solving this problem suggests a more general problem as follows. Given two polynomials P and Q, define the function

$$f(x) = \begin{cases} P(x), & \text{if } x \in \mathbb{Q}; \\ Q(x), & \text{if } x \notin \mathbb{Q}. \end{cases}$$

If a is a solution of the equation $P(x) = Q(x)$, i.e., $P(a) = Q(a)$, then the limit $\lim_{x \to a} f(x)$ exists and the limit is this common value. For all other points the limit does not exist.

Similar problems:

1. Determine all $a \in \mathbb{R}$ at which $\lim_{x\to a} f(x)$ exists, where

$$f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q}; \\ x+2, & \text{if } x \in x \notin \mathbb{Q}. \end{cases}$$

2. Consider the function

$$f(x) = \begin{cases} x^2+1, & \text{if } x \in \mathbb{Q}; \\ -x, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Prove that f does not have a limit at any $a \in \mathbb{R}$.

SECTION 3.2

Exercise 3.2.5. The given condition implies that if both x_1 and x_2 are close to $\bar{x}$, then they are close to each other and, hence, $f(x_1)$ and $f(x_2)$ are close to each other. This suggests the use of the Cauchy criterion for limit to solve the problem. Given any $\varepsilon > 0$, choose $\delta = \dfrac{\varepsilon}{2(k+1)}$. If $x_1, x_2 \in D \setminus \{\bar{x}\}$ with $|x_1 - \bar{x}| < \delta$ and $|x_2 - \bar{x}| < \delta$, then

$$|f(x_1) - f(x_2)| \le k|x_1 - x_2| \le k(|x_1 - \bar{x}| + |x_2 - \bar{x}|) < k(\delta + \delta) = 2k\frac{\varepsilon}{2(k+1)} < \varepsilon.$$

Therefore, $\lim_{x\to\bar{x}} f(x)$ exists.

SECTION 3.3

Exercise 3.3.8. (a) Observe that $f(a) = g(a) = h(a)$ and, hence,

$$|f(x) - f(a)| = \begin{cases} |g(x) - g(a)|, & \text{if } x \in \mathbb{Q} \cap [0,1]; \\ |h(x) - h(a)|, & \text{if } x \in \mathbb{Q}^c \cap [0,1]. \end{cases}$$

It follows that

$$|f(x) - f(a)| \le |g(x) - g(a)| + |h(x) - h(a)| \text{ for all } x \in [0,1].$$

Therefore, $\lim_{x\to a} f(x) = f(a)$ and, so, f is continuous at a.
(b) Apply part (a).

Exercise 3.3.9. At any irrational number $a \in (0,1]$, we have $f(a) = 0$. If x is near a and x is irrational, it is obvious that $f(x) = 0$ is near $f(a)$. In the case when x is near a and x is rational, $f(x) = 1/q$ where $p, q \in \mathbb{N}$. We will see in part (a) that for any $\varepsilon > 0$, there is only a finite number of $x \in (0,1]$ such that $f(x) \ge \varepsilon$. So $f(x)$ is close to $f(a)$ for all $x \in (0,1]$ except for a finite number of $x \in \mathbb{Q}$. Since a is irrational, we can choose a sufficiently small neighborhood of a to void such x.

(a) For any $\varepsilon > 0$,

$$A_\varepsilon = \{x \in (0,1] : f(x) \ge \varepsilon\} = \left\{x = \frac{p}{q} \in \mathbb{Q} : f(x) = \frac{1}{q} \ge \varepsilon\right\} = \left\{x = \frac{p}{q} \in \mathbb{Q} : q \le \frac{1}{\varepsilon}\right\}.$$

Clearly, the number of $q \in \mathbb{N}$ such that $q \le \frac{1}{\varepsilon}$ is finite. Since $0 < \frac{p}{q} \le 1$, we have $p \le q$. Therefore, A_ε is finite.

(b) Fix any irrational number $a \in (0,1]$. Then $f(a) = 0$. Given any $\varepsilon > 0$, by part (a), the set A_ε is finite, so we can write

$$A_\varepsilon = \{x \in (0,1] : f(x) \ge \varepsilon\} = \{x_1, x_2, \ldots, x_n\},$$

for some $n \in \mathbb{N}$, where $x_i \in \mathbb{Q}$ for all $i = 1, \ldots, n$. Since a is irrational, we can choose $\delta > 0$ such that $x_i \notin (a-\delta, a+\delta)$ for all $i = 1, \ldots, n$ (more precisely, we can choose $\delta = \min\{|a - x_i| : i = 1, \ldots, n\}$). Then

$$|f(x) - f(a)| = f(x) < \varepsilon \text{ whenever } |x - a| < \delta.$$

Therefore, f is continuous at a.

Now fix any rational number $b = \frac{p}{q} \in (0,1]$. Then $f(b) = \frac{1}{q}$. Choose a sequence of irrational numbers $\{s_n\}$ that converges to b. Since $f(s_n) = 0$ for all $n \in \mathbb{N}$, the sequence $\{f(s_n)\}$ does not converge to $f(b)$. Therefore, f is not continuous at b.

In this problem, we consider the domain of f to be the interval $(0,1]$, but the conclusion remain valid for other intervals. In particular, we can show that the function defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} \dfrac{1}{q}, & \text{if } x = \dfrac{p}{q}, p, q \in \mathbb{N}, \text{where } p \text{ and } q \text{ have no common factors;} \\ 1, & \text{if } x = 0; \\ 0, & \text{if } x \text{ is irrational,} \end{cases}$$

is continuous at every irrational point, and discontinuous at every rational point.

Exercise 3.3.10. Consider

$$f(x) = \begin{cases} (x - a_1)(x - a_2)\cdots(x - a_k), & \text{if } x \in \mathbb{Q}; \\ 0, & \text{if } x \in \mathbb{Q}^c. \end{cases}$$

SECTION 3.4

Exercise 3.4.6. Let $\alpha = \min\{f(x) : x \in [a,b]\}$ and $\beta = \max\{f(x) : x \in [a,b]\}$. Then

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \le \frac{n\beta}{n} = \beta.$$

Similarly,

$$\alpha \le \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

Then the conclusion follows from the Intermediate Value Theorem.

Exercise 3.4.7. (a) Observe that

$$|f(1/n)| \le 1/n \text{ for all } n \in \mathbb{N}.$$

(b) Apply the Extreme Value Theorem for the function $g(x) = \left|\frac{f(x)}{x}\right|$ on the interval $[a,b]$.

Exercise 3.4.8. First consider the case where f is monotone decreasing on $[0,1]$. By Exercise 3.4.5, f has a fixed point in $[0,1]$, which means that there exists $x_0 \in [0,1]$ such that

$$f(x_0) = x_0.$$

Since f is monotone decreasing, f has a unique fixed point. Indeed, suppose that there exists $x_1 \in [0,1]$ such that $f(x_1) = x_1$. If $x_1 < x_0$, then $x_1 = f(x_1) \geq f(x_0) = x_0$, which yields a contradiction. It is similar for the case where $x_1 > x_0$. Therefore, x_0 is the unique point in $[0,1]$ such that $f(x_0) = x_0$.

Since $f(g(x)) = g(f(x))$ for all $x \in [0,1]$, we have

$$f(g(x_0)) = g(f(x_0)) = g(x_0).$$

Thus, $g(x_0)$ is also a fixed point of f and, hence, $g(x_0) = x_0 = f(x_0)$. The proof is complete in this case.

Consider the case where f is monotone increasing. In this case, f could have several fixed points on $[0,1]$, so the previous argument does not work. However, by Exercise 3.4.5, there exists $c \in [0,1]$ such that $g(c) = c$. Define the sequence $\{x_n\}$ as follows:

$$\begin{aligned} x_1 &= c, \\ x_{n+1} &= f(x_n) \text{ for all } n \geq 1. \end{aligned}$$

Since f is monotone increasing, $\{x_n\}$ is a monotone sequence. In fact, if $x_1 \leq x_2$, then $\{x_n\}$ is monotone increasing; if $x_1 \geq x_2$, then $\{x_n\}$ is monotone decreasing. Since f is bounded, by the monotone convergence theorem (Theorem 2.3.1), there exists $x_0 \in [0,1]$ such that

$$\lim_{n\to\infty} x_n = x_0.$$

Since f is continuous and $x_{n+1} = f(x_n)$ for all $n \in \mathbb{N}$, taking limits we have $f(x_0) = x_0$.

We can prove by induction that $g(x_n) = x_n$ for all $n \in \mathbb{N}$. Then

$$g(x_0) = \lim_{n\to\infty} g(x_n) = \lim x_n = x_0.$$

Therefore, $f(x_0) = g(x_0) = x_0$.

SECTION 3.5

Exercise 3.5.2. (a) Let $f\colon D \to \mathbb{R}$. From Theorem 3.5.3 we see that if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in D such that $|x_n - y_n| \to 0$ as $n \to \infty$, but $\{|f(x_n) - f(y_n)|\}$ does not converge to 0, then f is not uniformly continuous on D. Roughly speaking, in order for f to be uniformly continuous on D, if x and y are close to each other, then $f(x)$ and $f(y)$ must be close to each other. The behavior of the graph of the squaring function suggests the argument below to show that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Define two sequences $\{x_n\}$ and $\{y_n\}$ as follows: $x_n = n$ and $y_n = n + \frac{1}{n}$ for $n \in \mathbb{N}$. Then $|x_n - y_n| = \frac{1}{n} \to 0$ as $n \to \infty$. However,

$$|f(x_n) - f(y_n)| = \left(n + \frac{1}{n}\right)^2 - n^2 = 2 + \frac{1}{n^2} \geq 2 \text{ for all } n \in \mathbb{N}.$$

Therefore, $\{|f(x_n) - f(y_n)|\}$ does not converge to 0 and, hence, f is not uniformly continuous on $\mathbb{R}$. In this solution, we can use $x_n = \sqrt{n + \frac{1}{n}}$ and $y_n = \sqrt{n}$ for $n \in \mathbb{N}$ instead.

(b) Use $x_n = \dfrac{1}{\pi/2 + 2n\pi}$ and $y_n = \dfrac{1}{2n\pi}, n \in \mathbb{N}$.

(c) Use $x_n = 1/n$ and $y_n = 1/(2n)$.

It is natural to ask whether the function $f(x) = x^3$ is uniformly continuous on $\mathbb{R}$. Following the solution for part (a), we can use $x_n = \sqrt[3]{n + \frac{1}{n}}$ and $y_n = \sqrt[3]{n}$ for $n \in \mathbb{N}$ to prove that f is not uniformly continuous on $\mathbb{R}$. By a similar method, we can show that the function $f(x) = x^n$, $n \in \mathbb{N}$, $n \geq 2$, is not uniformly continuous on $\mathbb{R}$. A more challenging question is to determine whether a polynomial of degree greater than or equal to two is uniformly continuous on $\mathbb{R}$.

Exercise 3.5.7. *Hint:* For part (a) use Theorem 3.5.5. For part (b) prove that the function can be extended to a continuous function on $[a,b]$ and then use Theorem 3.5.5.

Exercise 3.5.8. (a) Applying the definition of limit, we find $b > a$ such that

$$c - 1 < f(x) < c + 1 \text{ whenever } x > b.$$

Since f is continuous on $[a,b]$, it is bounded on this interval. Therefore, f is bounded on $[a,\infty)$.
(b) Fix any $\varepsilon > 0$, by the definition of limit, we find $b > a$ such that

$$|f(x) - c| < \frac{\varepsilon}{2} \text{ whenever } x > b.$$

Since f is continuous on $[a, b+1]$, it is uniformly continuous on this interval. Thus, there exists $0 < \delta < 1$ such that

$$|f(u) - f(v)| < \frac{\varepsilon}{2} \text{ whenever } |u - v| < \delta, u, v \in [a, c+1].$$

Then we can show that $|f(u) - f(v)| < \varepsilon$ whenever $|u - v| < \delta$, $u, v \in [a,\infty)$.
(c) Since $\lim_{x\to\infty} f(x) = c > f(a)$, there exists $b > a$ such that

$$f(x) > f(a) \text{ whenever } x > b.$$

Thus,

$$\inf\{f(x) : x \in [a,\infty)\} = \inf\{f(x) : x \in [a,b]\}.$$

The conclusion follows from the Extreme Value Theorem for the function f on $[a,b]$.

SECTION 3.6

Exercise 3.7.4. Since $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty$, there exists $a > 0$ such that

$$f(x) \geq f(0) \text{ whenever } |x| > a.$$

Since f is lower semicontinuous, by Theorem 3.7.3, it has an absolute minimum on $[-a,a]$ at some point $\bar{x} \in [-a,a]$. Obviously,

$$f(x) \geq f(\bar{x}) \text{ for all } x \in [-a,a].$$

In particular, $f(0) \geq f(\bar{x})$. If $|x| > a$, then

$$f(x) \geq f(0) \geq f(\bar{x}).$$

Therefore, f has an absolute minimum at $\bar{x}$.

Observe that in this solution, we can use any number γ in the range of f instead of $f(0)$. Since any continuous function is also lower semicontinuous, the result from this problem is applicable for continuous functions. For example, we can use this theorem to prove that any polynomial with even degree has an absolute minimum on $\mathbb{R}$. Since $\mathbb{R}$ is a not a compact set, we cannot use the extreme value theorem directly.

SECTION 4.1

Exercise 4.1.10. Use the identity

$$\lim_{n\to\infty}\left(\frac{f(a+\frac{1}{n})}{f(a)}\right)^n = \lim_{n\to\infty}\exp(n[\ln(f(a+\frac{1}{n})) - \ln(f(a)]).$$

Exercise 4.1.11. (a) Using the differentiability of $\sin x$ and Theorem 4.1.3, we conclude the function is differentiable at any $a \neq 0$. So, we only need to show the differentiability of the function at $a = 0$. By the definition of the derivative, consider the limit

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to 0}\frac{x^2\sin(1/x)+cx}{x} = \lim_{x\to 0}[x\sin(1/x)+c].$$

For any $x \neq 0$, we have

$$|x\sin(1/x)| = |x|\,|\sin(1/x)| \leq |x|,$$

which implies

$$-|x| \leq x\sin(1/x) \leq |x|.$$

Since $\lim_{x\to 0}(-|x|) = \lim_{x\to 0}|x| = 0$, applying the squeeze theorem yields

$$\lim_{x\to 0} x\sin(1/x) = 0.$$

It now follows that

$$f'(0) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to 0}[x\sin(1/x)+c] = c.$$

Using Theorem 4.1.3 and the fact that $\cos x$ is the derivative of $\sin x$, the derivative of f can be written explicitly as

$$f'(x) = \begin{cases} 2x\sin\dfrac{1}{x} - \cos(1/x) + c, & \text{if } x \neq 0; \\ c, & \text{if } x = 0. \end{cases}$$

From the solution, it is important to see that the conclusion remains valid if we replace the function f by

$$g(x) = \begin{cases} x^n \sin\dfrac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0, \end{cases}$$

where $n \geq 2$, $n \in \mathbb{N}$. Note that the function $h(x) = cx$ does not play any role in the differentiability of f.

We can generalize this problem as follows. Let φ be a bounded function on $\mathbb{R}$, i.e., there is $M > 0$ such that

$$|\varphi(x)| \leq M \text{ for all } x \in \mathbb{R}.$$

Define the function

$$f(x) = \begin{cases} x^n \varphi(1/x), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0, \end{cases}$$

where $n \geq 2$, $n \in \mathbb{N}$. Then f is differentiable at $a = 0$.

Similar problems:

1. Show that the functions below are differentiable on $\mathbb{R}$:

$$f(x) = \begin{cases} x^{3/2} \cos(1/x), & \text{if } x \geq 0; \\ 0, & \text{if } x < 0 \end{cases}$$

and

$$f(x) = \begin{cases} x^2 e^{-1/x^2}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

2. Suppose that φ is bounded and differentiable on $\mathbb{R}$. Define the function

$$f(x) = \begin{cases} x^n \varphi(1/x), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Show that if $n \geq 2$, the function is differentiable on $\mathbb{R}$ and find its derivative. Show that if $n = 1$ and $\lim_{x\to\infty} \varphi(x)$ does not exists, then f is not differentiable at 0.

(b) Hint: Observe that

$$f'\left(\frac{1}{2n\pi}\right) = -1 + c < 0 \text{ and } f'\left(\frac{1}{(2n+1)\pi}\right) = 1 + c > 0.$$

SECTION 4.2

Exercise 4.2.1. Define the function

$$h(x) = f(x) - g(x).$$

Then h has an absolute maximum at x_0. Thus,

$$h'(x_0) = f'(x_0) - g'(x_0) = 0,$$

which implies $f'(x_0) = g'(x_0)$.

Exercise 4.2.3. The inequality holds obviously if $a = b$. In the case where $a \neq b$, the equality can be rewritten as

$$\left| \frac{\sin(b) - \sin(a)}{b - a} \right| \leq 1.$$

The quotient $\left| \frac{\sin(b) - \sin(a)}{b - a} \right|$ is the slope of the line connecting $(a, f(a))$ and $(b, f(b))$. We need to show that the absolute value of the slope is always bounded by 1, which can also be seen from the figure. The quotient also reminds us of applying the Mean Value Theorem for the function $f(x) = \sin(x)$.

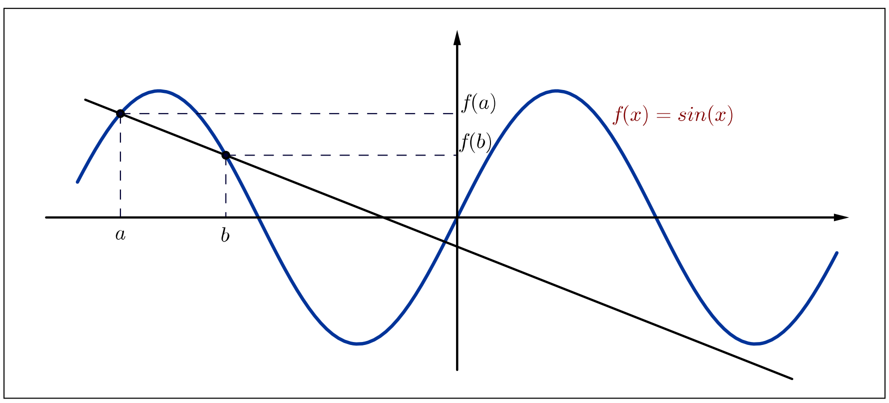

Figure 5.1: The function $f(x) = \sin(x)$.

Consider the case where $a < b$ and define the function $f\colon [a, b] \to \mathbb{R}$ by $f(x) = \sin(x)$. Clearly, the function satisfies all assumptions of the Mean Value Theorem on this interval with $f'(x) = \cos(x)$ for all $x \in (a, b)$.

By the Mean Value Theorem, there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) = \cos(c),$$

which implies

$$\left| \frac{f(b) - f(a)}{b - a} \right| = |\cos(c)| \leq 1.$$

It follows that $|f(a) - f(b)| \leq |a - b|$. The solution is similar for the case where $a > b$.

It is essential to realize that the most important property required in solving this problem is the boundedness of the derivative of the function. Thus, it is possible to solve the following problems

with a similar strategy.
1. Prove that $|\cos(a)-\cos(b)| \le |a-b|$ for all $a,b \in \mathbb{R}$.
2. Prove that $|\ln(1+e^{2a})-\ln(1+e^{2b})| \le 2|a-b|$ for all $a,b \in \mathbb{R}$.

Exercise 4.2.4. Let us define $f\colon [-\pi,\pi] \to \mathbb{R}$ by

$$f(x) = x + \sum_{k=1}^{n} (a_k \sin kx + b_k \cos kx).$$

We want to find $c \in (-\pi,\pi)$ such that $f(c) = 0$.

Now, consider the function

$$g(x) = \frac{x^2}{2} + \sum_{k=1}^{n} \left(-a_k \frac{\cos(kx)}{k} + b_k \frac{\sin(kx)}{k} \right).$$

Observe that $g(-\pi) = g(\pi)$ and $g' = f$. The conclusion follows from Rolle's Theorem.

Exercise 4.2.5. Use the identity

$$\frac{1}{g(b)-g(a)} \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = \frac{f(a)g(b)-f(b)g(a)}{g(b)-g(a)} = \frac{\frac{f(a)}{g(a)} - \frac{f(b)}{g(b)}}{\frac{1}{g(a)} - \frac{1}{g(b)}}.$$

Then apply the Cauchy mean value theorem for two functions $\phi(x) = \frac{f(x)}{g(x)}$ and $\psi(x) = \frac{1}{g(x)}$ on the interval $[a,b]$.

Exercise 4.2.6. 1. Apply Rolle's theorem to the function

$$f(x) = a_1 x + a_2 \frac{x^2}{2} + \cdots + a_n \frac{x^n}{n}$$

on the interval $[0,1]$.
2. Apply Rolle's theorem to the function

$$f(x) = \sum_{k=0}^{n} \frac{\sin(2k+1)x}{2k+1}$$

on the interval $[0,\pi/2]$.

Exercise 4.2.8. (a) Given $\varepsilon > 0$, first find x_0 large enough so that $a-\varepsilon/2 < f'(x) < a+\varepsilon/2$ for $x > x_0$. Then use the identity

$$\frac{f(x)}{x} = \frac{f(x)-f(x_0)+f(x_0)}{x-x_0+x_0} = \frac{\frac{f(x)-f(x_0)}{x-x_0} + \frac{f(x_0)}{x-x_0}}{1+\frac{x_0}{x-x_0}},$$

and the mean value theorem to show that, for x large,

$$a-\varepsilon < \frac{f(x)}{x} < a+\varepsilon.$$

(b) Use the method in part (a).
(c) Consider $f(x) = \sin(x)$.

SECTION 4.3

Exercise 4.3.2. (a) We can prove that f is uniformly continuous on $\mathbb{R}$ by definition. Given any $\varepsilon > 0$, choose $\delta = \left(\frac{\varepsilon}{\ell+1}\right)^{\frac{1}{\alpha}}$ and get

$$|f(u) - f(v)| \leq \ell|u-v|^{\alpha} < \ell\delta^{\alpha} = \ell\frac{\varepsilon}{\ell+1} < \varepsilon$$

whenever $|u-v| < \delta$. Note that we use $\ell + 1$ here instead of ℓ to avoid the case where $\ell = 0$.

(b) We will prove that f is a constant function by showing that it is differentiable on $\mathbb{R}$ and $f'(a) = 0$ for all $a \in \mathbb{R}$. Fix any $a \in \mathbb{R}$. Then, for $x \neq a$,

$$\left|\frac{f(x)-f(a)}{x-a}\right| \leq \frac{\ell|x-a|^{\alpha}}{|x-a|} = \ell|x-a|^{\alpha-1}.$$

Since $\alpha > 1$, by the squeeze theorem,

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = 0.$$

This implies that f is differentiable at a and $f'(a) = 0$.

(c) We can verify that the function $f(x) = |x|$ satisfies the requirement.

From this problem, we see that it is only interesting to consider the class of functions that satisfy (4.8) when $\alpha \leq 1$. It is an exercise to show that the function $f(x) = |x|^{1/2}$ satisfies this condition with $\ell = 1$ and $\alpha = 1/2$.

Exercise 4.3.3. Define the function

$$h(x) = g(x) - f(x).$$

Then $h'(x) = g'(x) - f'(x) \geq 0$ for all $x \in [x_0, \infty)$. Thus, h is monotone increasing on this interval. It follows that

$$h(x) \geq h(x_0) = g(x_0) - f(x_0) = 0 \text{ for all } x \geq x_0.$$

Therefore, $g(x) \geq f(x)$ for all $x \geq x_0$.

Exercise 4.3.5. Apply the mean value theorem twice.

Exercise 4.3.6. Use proof by contradiction.

SECTION 4.4

Exercise 4.4.5. Suppose that

$$P(x) = a_0 + a_1x + \cdots + a_nx^n.$$

Then apply L'Hospital's rule repeatedly.

Exercise 4.4.6. We first consider the case where $n = 1$ to get ideas for solving this problem in the general case. From the standard derivative theorems we get that the function is differentiable at any $x \neq 0$ with

$$f'(x) = 2x^{-3}e^{-\frac{1}{x^2}} = \frac{2}{x^3}e^{-\frac{1}{x^2}}.$$

Consider the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x}.$$

Letting $t = 1/x$ and applying L'Hospital rule yields

$$\lim_{x\to 0^+} \frac{e^{-\frac{1}{x^2}}}{x} = \lim_{t\to\infty} \frac{t}{e^{t^2}} = \lim_{t\to\infty} \frac{1}{2te^{t^2}} = 0.$$

Similarly,

$$\lim_{x\to 0^-} \frac{e^{-\frac{1}{x^2}}}{x} = 0.$$

It follows that f is differentiable on $\mathbb{R}$ with

$$f'(x) = \begin{cases} \frac{2}{x^3}e^{-\frac{1}{x^2}}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

In a similar way, we can show that f is twice differentiable on $\mathbb{R}$ with

$$f''(x) = \begin{cases} \left(\frac{6}{x^4} + \frac{2}{x^6}\right)e^{-\frac{1}{x^2}}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Based on these calculations, we predict that f is n times differentiable for every $n \in \mathbb{N}$ with

$$f^{(n)}(x) = \begin{cases} P\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0, \end{cases}$$

where P is a polynomial. Now we proceed to prove this conclusion by induction. The conclusion is true for $n = 1$ as shown above. Given that the conclusion is true for some $n \in \mathbb{N}$, for $x \neq 0$ we have

$$f^{(n+1)}(x) = -x^{-2}P'\left(\frac{1}{x}\right) + \frac{2}{x^3}P\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}} = Q\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}},$$

where Q is also a polynomial. It is an easy exercise to write the explicit formula of Q based on P. Moreover, successive applications of l'Hôpital's rule give

$$\lim_{x\to 0^+} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x\to 0^+} \frac{1}{x}P\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}} = \lim_{t\to\infty} \frac{tP(t)}{e^{t^2}} = 0.$$

In a similar way, we can show that

$$\lim_{x\to 0^-} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = 0.$$

Therefore, $f^{(n+1)}(0) = 0$. We have proved that for every $n \in \mathbb{N}$, f is n times differentiable and, so, $f \in C^n(\mathbb{R})$. Here we do not need to prove the continuity of $f^{(n)}$ because the differentiability of $f^{(n)}$ implies its continuity.

In a similar way, we can also show that the function

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & \text{if } x > 0; \\ 0, & \text{if } x \leq 0 \end{cases}$$

is n times differentiable for every $n \in \mathbb{N}$.

SECTION 4.5

Exercise 4.5.1. Let $f(x) = e^x$. By Taylor's theorem, for any $x > 0$, there exists $c \in (0, x)$ such that

$$\begin{aligned} f(x) = e^x &= \sum_{k=0}^{m} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(m+1)}(c)}{(m+1)!} c^{m+1} \\ &= \sum_{k=0}^{m} \frac{x^k}{k!} + \frac{e^c}{(m+1)!} c^{m+1} > \sum_{k=0}^{m} \frac{x^k}{k!}. \end{aligned}$$

Exercise 4.5.5. (a) Observe that a simpler version of this problem can be stated as follows: If f is differentiable on (a,b) and $\bar{x} \in (a,b)$, then

$$\lim_{h \to 0} \frac{f(\bar{x}+h) - f(\bar{x})}{h} = \frac{f'(\bar{x})}{1!}.$$

This conclusion follows directly from the definition of derivative.

Similarly, if f is twice differentiable on (a,b) and $\bar{x} \in (a,b)$, then

$$\lim_{h \to 0} \frac{f(\bar{x}+h) - f(\bar{x}) - f'(\bar{x})\frac{h}{1!}}{h^2} = \frac{f''(\bar{x})}{2!}.$$

We can prove this by applying the L'Hospital rule to get

$$\lim_{h \to 0} \frac{f(\bar{x}+h) - f(\bar{x}) - f'(\bar{x})\frac{h}{1!}}{h^2} = \lim_{h \to 0} \frac{f'(\bar{x}+h) - f'(\bar{x})}{2h} = \frac{f''(\bar{x})}{2!}.$$

It is now clear that we can solve part (a) by using the L'Hospital rule as follows:

$$\lim_{h \to 0} \frac{f(\bar{x}+h) - f(\bar{x}) - f'(\bar{x})\frac{h}{1!} - f''(\bar{x})\frac{h^2}{2!}}{h^3} = \lim_{h \to 0} \frac{f'(\bar{x}+h) - f'(\bar{x}) - f''(\bar{x})\frac{h}{1!}}{3h^2} = \frac{f'''(\bar{x})}{3!}.$$

Note that the last equality follows from the previous proof applied to the function f'.

(b) With the analysis from part (a), we see that if f is n times differentiable on (a,b) and $\bar{x} \in (a,b)$, then

$$\lim_{h \to 0} \frac{f(\bar{x}+h) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\bar{x})h^k}{k!}}{h^{n+1}} = \frac{f^{(n)}(\bar{x})}{n!}.$$

This conclusion can be proved by induction. This general result can be applied to obtain the *Taylor expansion with Peano's remainder* in Exercise 4.5.6.

SECTION 4.6

Exercise 4.6.3. We apply the definition to solve this problem. Given any $u, v \in I$ and $\lambda \in (0,1)$, we have

$$f(\lambda u + (1-\lambda)v) \leq \lambda f(u) + (1-\lambda)f(v)$$

by the convexity of f.

Since $f(u), f(v) \in J$ and J is an interval, $\lambda f(u) + (1-\lambda)f(v) \in J$. By the nondecreasing property and the convexity of ϕ,

$$\phi(f(\lambda u + (1-\lambda)v)) \leq \phi(\lambda f(u) + (1-\lambda)f(v)) \leq \lambda \phi(f(u)) + (1-\lambda)\phi(f(v)).$$

Therefore, $\phi \circ f$ is convex on I.

The result from this problem allows us to generate convex functions. For example, consider $f(x) = |x|$ and $\phi(x) = x^p$, $p > 1$. We have seen that f is convex on $\mathbb{R}$. The function ϕ is convex and increasing on $[0, \infty)$ which contains the range of the function f. Therefore, the composition $g(x) = |x|^p$, $p > 1$, is convex on $\mathbb{R}$. Similarly, $h(x) = e^{x^2}$ is also a convex function on $\mathbb{R}$.

Observe that in this problem, we require the nondecreasing property of ϕ. A natural question is whether the composition of two convex functions is convex. The answer is negative. Observe that $f(x) = x^2$ and $\phi(x) = |x-1|$ are convex, but $(\phi \circ f)(x) = |x^2 - 1|$ is nonconvex.

Exercise 4.6.4. Use Theorem 4.6.6 or Corollary 4.6.7.

Exercise 4.6.5. (a) Use the obvious inequality

$$(\sqrt{a} - \sqrt{b})^2 \geq 0.$$

Alternatively, consider the function $f(x) = -\ln(x)$, $x \in (0, \infty)$. We can show that f is convex on $(0, \infty)$. For $a, b \in (0, \infty)$, one has

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}.$$

This implies

$$-\ln(\frac{a+b}{2}) \leq \frac{-\ln(a) - \ln(b)}{2} = -\ln(\sqrt{ab}).$$

Therefore,

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

This inequality holds obviously when $a = 0$ or $b = 0$.
(b) Use Theorem 4.6.3 for the function $f(x) = -\ln(x)$ on $(0, \infty)$.

SECTION 4.7

Exercise 4.7.1. (a) By Theorem 4.7.5,

$$\partial f(x) = \begin{cases} \{-a\}, & \text{if } x < 0; \\ [-a, a], & \text{if } x = 0; \\ \{a\}, & \text{if } x > 0. \end{cases}$$

(b) By Theorem 4.7.5,

$$\partial f(x) = \begin{cases} \{-2\}, & \text{if } x < -1; \\ [-2,0], & \text{if } x = -1; \\ \{0\}, & \text{if } x \in (-1,1); \\ [0,2], & \text{if } x = 1; \\ \{2\}, & \text{if } x > 1. \end{cases}$$

Exercise 4.7.3. To better understand the problem, we consider some special cases. If $n = 1$, then $f(x) = |x-1|$. Obviously, f has an absolute minimum at $x = 1$. If $n = 2$, then $f(x) = |x-1| + |x-2|$. The graphing of the function suggests that f has an absolute minimum at any $x \in [1,2]$. In the case where $n = 3$, we can see that f has an absolute minimum at $x = 2$. We then conjecture that if n is odd with $n = 2m-1$, then f has an absolute minimum at $x = m$. If n is even with $n = 2m$, then f has an absolute minimum at any point $x \in [m, m+1]$.

Let us prove the first conclusion. In this case,

$$f(x) = \sum_{i=1}^{2m-1} |x-i| = \sum_{i=1}^{2m-1} f_i(x),$$

where $f_i(x) = |x-i|$. Consider $\bar{x} = m$. Then

$$\partial f_m(\bar{x}) = [-1,1],\ \partial f_i(\bar{x}) = \{1\} \text{ if } i < m,\ \partial f_i(\bar{x}) = \{-1\} \text{ if } i > m.$$

The subdifferential sum rule yields $\partial f(\bar{x}) = [-1,1]$ which contains 0. Thus, f has an absolute minimum at $\bar{x}$. If $\bar{x} > m$, we can see that $\partial f(\bar{x}) \subset (0,\infty)$, which does not contain 0. Similarly, if $\bar{x} < m$, then $\partial f(\bar{x}) \subset (-\infty,0)$. Therefore, f has an absolute minimum at the only point $\bar{x} = m$.

The case where n is even can be treated similarly.

Exercise 4.7.5. Fix $a, b \in \mathbb{R}$ with $a < b$. By Theorem 4.7.9, there exists $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} \in \partial f(c) \subset [0,\infty).$$

This implies $f(b) - f(a) \geq 0$ and, hence, $f(b) \geq f(a)$. Therefore, f is monotone increasing on $\mathbb{R}$,

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